The Dynamics of Social Instability

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February 4, 2022

Abstract

We study a model in which two groups with opposing interests try to alter a status quo through instability-generating actions. We show that even if these actions are costly and do not lead to any short-term average utility gains, they can be used to secure longer-term durable changes. In equilibrium, the level of instability generated by a group decreases when the status quo favors it more. Equilibrium always exhibits stable states in which the status quo persists indefinitely. There is long-run path-dependency and inequity: although the process of social instability eventually leads to a stable state, it will typically select the least favorable one for the initially disadvantaged group.

Keywords: Instability; Social Conflict; Stochastic Games.
JEL Classifications: C72, C73, C78, D74.

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We thank Yeon-Koo Che, Joan Esteban, Navin Kartik, Elliot Lipnowski, and Yu Fu Wong for helpful comments and discussions.

Instability is an essential component of demonstrations and protests, riots and civil disobedience, nonviolent resistance and violent unrest, outrageous stances and fiery speeches. The women's suffrage movement in the early 20th century Britain relied not only on parades but also arson and other instances of violent unrest; the civil rights movement in the 1950-60s United States was marked by nonviolence resistance campaigns; a number of other contemporary movements — Occupy Wall Street, Black Lives Matter demonstrations, environmental activism — create instability to induce social change. Groups with diametrically opposed interests use instability to advance their agenda; this is especially salient in examples such as abortion rights or political campaigns. Sometimes actions have no predictable effect other than to increase instability, and instability itself becomes a means to an end: social unrest generated by a group can have unpredictable effects and be equally likely to advance their cause among the public or to backfire; a politician's bold claims might win or lose support with equal probability. How can instability emerge as a valuable tool for groups with conflicting interests aiming to secure durable changes on the status quo when it offers no short-term advantage? Does social instability give rise to a stable state, a status quo that no group wishes to destabilize, or is it instead self-perpetuating?

In this paper, we study the strategic implications of using social instability as an instrument for groups to change a status quo. We consider a model in which two forward-looking players accrue bounded constant-sum gross flow payoffs, representing groups with diametrically opposed interests over a given state. At any moment, players can pay a cost to increase the volatility of a process that determines the status quo, capturing the use of instability as a means of social conflict. In particular, instability has a symmetric effect everywhere but at the extreme states, where it can only reflect the process towards less extreme states.

We show that instability can be an effective device to promote change. If the only thing that a player can do is to destabilize the status quo in a way that change is equally likely to be favorable or unfavorable, instability cannot offer any advantage in the short-term. Furthermore, even if instability were to lead to a more favorable situation for the player, it could be met with additional instability by others with opposing views, further depressing the incentives to take action. But when a player has nothing to lose, instability seems like a natural instrument to oppose an excessively unfavorable status quo. We show how a lower bound on the negative consequences of creating instability provides option value that can be exploited by patient players, even when they stand to lose as much as they stand to gain in the short run.

We identify two key properties of players' optimal (Markovian) behavior. First, an optimal volatility strategy in response to any strategy of the opponent is characterized by a threshold mechanism: players continuously generate positive volatility at situations less favorable than a target "satisficing" state, and no instability at more favorable ones. Second, optimal strategies are monotone, creating more instability at less favorable states. Because gains over the status quo are driven by the option value conferred by the lower bound on how unfavorable the state can be, at a more favorable status quo, this option value decreases, and players become more conservative as they stand to lose more.

We then prove existence and provide a complete constructive characterization of the set of equilibria. An intuitive decoupling argument lies at the heart of this characterization: at most one player creates instability at any given moment. As instability yields no short-term gains on the status quo and players have diametrically opposed interests, they cannot both expect to benefit from it. As a result, equilibria are completely characterized by two thresholds, defining two regions of instability, and a stable region wherein the status quo prevails. Instability arises in the most extreme states, and the player who least favors the status quo creates instability to strive for change. Social instability is used as a tool to push back against extreme situations, and more extreme situations foster greater instability. In contrast, in the stable region — corresponding to relatively more moderate states — neither player sees advantage in destabilizing the status quo.

While equilibrium stable states always exist, these can be either the expression of accommodating equilibrium behavior, or of a balance-of-power mechanism. In the former, players never push back against instability triggered by their opponent and so each player pursues gains on the status quo by generating instability exactly as if they faced no opposition. Such accommodating behavior occurs when impatience and costs to instability are high enough, which, owing to the threshold structure of best responses, supports a unique equilibrium. This unique equilibrium generically features a continuum of stable states: those that are satisfactory to both players and, if perturbed, would not trigger any instability. The situation fully reverses when players are patient and costs to instability are low enough: multiple equilibria arise, and each is characterized by a unique stable state. Further, this unique stable status quo emerges as resulting from players actively pushing back against their opponent's attempts to advance their agenda. Equilibrium behavior is then characterized as a balance-of-power mechanism at the stable status quo: the knowledge that the opponent will trigger social instability if the status quo is perturbed to their detriment deters the player from pursuing further improvements.

Equilibria also exhibit clear monotone comparative statics. We find that lowering a player's costs to creating instability shifts the set of stable states in a strong set order sense toward states the player prefers, as the player is willing to generate more instability to pursue its goals. For instance, legal restrictions to demonstrations or even hefty legal penalties for unrest negatively affect the willingness of groups to create instability and thus favor the persistence of the status quo, while social media can be seen as effectively reducing the costs to groups coordinating on promoting instability-generating actions.

Finally, we discuss the dynamics of social instability in our model. We show that, regardless of the starting point, the process converges almost surely to an equilibrium stable state. Nevertheless, we note a form of path dependency: if the process starts in a player's instability region, it will converge to that player's least favorable stable state.

Our paper contributes to the literature studying theoretical models of conflict. Paraphrasing Fearon (1995, p. 387), conflict is "a gamble whose outcome may be determined by random or otherwise unforeseeable events." This observation motivated the modeling of conflict using contests, that is, situations in which players exert costly effort to affect their relative likelihood of obtaining a more favorable outcome. Starting with the seminal work of Tullock (1980) in studying political party competition, several papers use this modeling device to study issues related to conflict and competition, including conflict over the appropriation of rents (Besley and Persson 2011; Powell 2013), lobbying (Baye et al. 1993; Che and Gale 1998), territorial expansion (Bueno de Mesquita 2020; Dziubiński et al. 2021), and how inequality affects the intensity of social conflict (Esteban and Ray 1999, 2011). Our main contribution relative to the existing literature on conflict is to introduce a novel instability mechanism and relate it to key concepts and phenomena in the dynamics of conflict. Our model gives qualitatively reasonable predictions for the dynamics of instability and, in doing so, highlights that instability need not be a purely exogenous byproduct, but rather a powerful and important instrument in social conflict.

Social instability gives rise to two phenomena typically present in other models of conflict. First, the fact that disadvantaged groups are exactly the ones that trigger social instability is reminiscent of the idea that too unequal outcomes will trigger conflict (Fearon 1995). Second, although modeled in a different manner in either Jackson and Morelli (2009) or Chassang and Padró i Miquel (2010), the common theme of deterrence appears in our model in instances where a single state emerges as stable in equilibrium, and its stability is supported only by the fact that each of the two groups with opposing interests would escalate social conflict were the status quo affected.

Another related strand of the literature pertains to tug-of-war models. In particular, our paper is closest to Moscarini and Smith's (2011) continuous-time analogue of the model by Harris and Vickers (1987), in which players with antagonistic preferences exert effort to increase the probability a state moves toward their preferred outcome. We highlight three key differences with respect to our model: (i) gross payoffs are constant sum but only arise when the the state reaches one of two extremes, and, most importantly, (ii) the two extreme states are absorbing rather than reflecting, and (iii) players control the drift of the process rather the volatility of the process. Consequently, in stark contrast to our model, it is the player in the most advantageous position who exerts the most effort.

Lastly, our paper contributes to a growing literature on games in continuous time. Although the use of differential methods for zero-sum games dates back to the seminal work of Isaacs (1965), a number of recent contributions have effectively used stochastic calculus and differential equations techniques in continuous time games.[1] As other recent papers in economics (e.g. Faingold and Sannikov 2011; Kaplan et al. 2018; Achdou et al. 2021; Lester 2020; Kuvalekar and Lipnowski 2020; Escudé and Sinander 2021) and a wealth of applications in finance,[2] we rely on viscosity solutions to solve a non-smooth optimal control problem. Building on Lions's (1986), this paper provides a technical contribution to this literature by proving existence and uniqueness of optimal control of volatility of a process with reflecting boundary conditions under relaxed regularity conditions. We hope

[1]See Sannikov (2007) for applications to repeated games, or Daley and Green (2012; 2020) and Ortner (2019) for applications to bargaining with a continuous inflow of news and evolving bargaining power.

[2]See the monographs by Fleming and Soner (2006) or Pham (2009) for more detail.

that the present paper also serves to illustrate the usefulness of this approach for obtaining precise characterizations in economic applications while imposing minimal assumptions.

The remainder of the paper is organized as follows: Section 1 introduces the model. In Section 2, we give a detailed characterization of optimal instability strategies by studying the best response to a fixed opponent strategy; we pay particular attention the benchmark case when the opponent is inactive and a single player controls the volatility. We use these results in Section 3 to construct and characterize equilibria, and, in Section 4, we discuss the equilibrium dynamics of the status quo: namely, convergence towards a stable state. Section 5 discusses some natural variations of our model.

1. The Model

We now introduce our model. Time is continuous and indexed by $t \in \mathbb{R}_+$. The state at time t is given by $X_t \in [0,1]$, corresponding to a status quo, and players A and B represent two groups with opposing preferences over the status quo, captured by constant-sum flow payoffs. Player A strictly prefers higher values of the status quo, whereas B favors lower ones, and we remove any intrinsic incentive to generate instability by considering risk neutral preferences. Given these assumptions, it is without loss to normalize player A's gross payoff at time t to be given by X_t and player B's gross payoff by $1-X_t$.

The state evolves randomly and continuously over time according to the following stochastic differential equation with reflection:

$$dX_t = \sqrt{2(\alpha_t + \beta_t)}dB_t - dK_t,$$

where B_t is a standard Brownian motion, $\alpha_t \geq 0$ and $\beta_t \geq 0$ are non-negative adapted processes controlled by players A and B respectively, and K_t denotes the regulator process that prevents X_t from exiting the interval $[0,1]$.[3] Specifically, K_t reflects the process within $[0,1]$ when it hits either bound and is inactive in the interior — i.e. if $X_t \in (0,1)$ we have $dK_t = 0$.

[3]The presence of the regulator process K_t is purely a technical device used to define a process whose infinitesimal variations essentially follows $dX_t = \sqrt{2(\alpha_t+\beta_t)}dB_t$ but where an inward push compensates every variation that would push the process outside of the bounded domain $[0,1]$; K_t precisely defines this compensation to ensure that we have defined a process over $[0,1]$. We give more technical details on the definition of the process in Appendix A.

This captures the idea that instability has a symmetric effect everywhere but at the boundary. Over a small time interval, the change in the status quo is exclusively driven by instability: at any instant, X_t goes either up or down with equal probability, except at the boundaries (0 and 1), where it simply cannot become more extreme. Everywhere in the interior, the status quo changes in a purely noisy manner. We assume away any trend in order to isolate the effect of instability. While continuity of the process is essential in capturing the desired intuition — because over a small time interval the probability of hitting either bounds is zero, a form of local symmetry is ensured — the fact that players control the level of instability means that the state can change extremely quickly, or not at all.

Players A and B respectively control α_t and β_t — how much effort each puts into destabilizing the status quo. Total instability effort $\alpha_t + \beta_t$ is aggregated additively and corresponds to scaling the volatility of the Brownian motion, which is captured by the square root transformation $\sqrt{2(\alpha_t + \beta_t)}$ (the factor of 2 is just a convenient normalization without loss). Instantaneous volatility here is the continuous-time analogue of increasing variance in a discrete-time setting. In other words, players are always able to escalate social instability, but they cannot decrease instability triggered by the opposing group.

Observe that instability here is entirely endogenous: players can remain at the current status quo forever if they choose not to increase volatility ($\alpha_t = \beta_t = 0$), but each player has the ability to unilaterally generate instability. In this sense, a state X_t at which no player has an incentive to generate instability corresponds to a stable status quo. We focus on the stylized case in which all instability is endogenous to clearly identify its idiosyncratic effects.

Creating instability is costly. We assume the cost of instability effort is convex and adopt a quadratic specification for simplicity. The instantaneous (net) payoffs of A and B are respectively:

$$u_a(X_t, \alpha_t) := X_t - \frac{c_a}{2}\alpha_t^2, \qquad u_b(X_t, \beta_t) := (1 - X_t) - \frac{c_b}{2}\beta_t^2,$$

where $c_a, c_b \in \mathbb{R}_{++}$ are idiosyncratic cost parameters for each player. Because creating instability is costly, this is not a zero-sum game. At a given instant, instability requires a pure destruction of surplus which can only be warranted by the hope of obtaining a durably better situation in some appropriate sense.

We consider the game in which each player chooses its instability effort over an infinite horizon. Players have possibly heterogeneous discount factors r_a and r_b respectively and we normalize flow payoffs by the discount factors. Expected utilities as a function of strategies and the status quo (the initial point of the process) are given by:

$$U_a(\alpha,\beta \mid x) := \mathbb{E}\left[\int_0^\infty r_a e^{-r_a t} u_a(X_t,\alpha_t) d_t\right], \quad U_b(\alpha,\beta \mid x) := \mathbb{E}\left[\int_0^\infty r_b e^{-r_b t} u_b(X_t,\beta_t) d_t\right]$$

where $dX_t = \sqrt{2(\alpha_t+\beta_t)}dB_t - dK_t$ and $X_0 = x$.

We restrict attention to Markov-perfect equilibria (Maskin and Tirole 2001) in continuous strategies. We then denote strategies as $\alpha_t = a(X_t)$, $\beta_t = b(X_t)$, where a and b are continuous functions from $[0,1]$ to $\mathbb{R}_+$. Formally, strategies belong to the class of X_t-adapted progressively measurable processes, which we denote by $\mathscr{A}$. The restriction to Markov-perfect equilibria is common in the literature, due in part to well-known issues in defining off-path behavior in continuous time (see Simon and Stinchcombe 1989). Continuity is partly a technical assumption, albeit a natural one in our setup. It is also minimal in that it requires little regularity to ensure that the underlying objects are properly defined. We formally define our equilibrium concept:

Definition 1. An equilibrium is a pair of continuous functions (a,b) from $[0,1]$ to $\mathbb{R}_+$ such that:

(i) The process $\alpha_t^* = a(X_t)$ solves the control problem for player A given b:

$$\alpha^* \in \underset{\alpha\in\mathscr{A}}{\arg\max}\, \mathbb{E}\left[\int_0^\infty r_a e^{-r_a t}\left(X_t - \frac{c_a}{2}\alpha_t^2\right) d_t\right] \quad \text{s.t. } dX_t = \sqrt{2(\alpha_t + b(X_t))}dB_t - dK_t,\ X_0 = x.$$

(ii) The process $\beta_t^* = b(X_t)$ solves the control problem for player B given a:

$$\beta^* \in \underset{\beta\in\mathscr{A}}{\arg\max}\, \mathbb{E}\left[\int_0^\infty r_b e^{-r_b t}\left((1-X_t) - \frac{c_b}{2}\beta_t^2\right) d_t\right] \quad \text{s.t. } dX_t = \sqrt{2(a(X_t)+\beta_t)}dB_t - dK_t,\ X_0 = x.$$

In the next section, we study the control problem in detail for a fixed strategy of the opponent so as to characterize best responses in this game. In doing so, we will verify that the previous definition of equilibrium is appropriate; in particular, optimal strategies are well-defined and continuous. This also allows us to identify relevant properties of best responses, which will prove useful to provide a direct construction of equilibria in Section 3.

2. Characterizing Best Responses

In this section, we study the properties of players' best responses through its differential characterization. We consider the control problem of one player, holding fixed the strategy of the opponent. Since the individual problems of the players are symmetric by definition when replacing X_t by $1-X_t$ in the flow payoff, we will consider player A's problem. All results extend symmetrically to player B's problem (see Section 2.4). As we focus on player A's problem, throughout this section we will omit the a subscripts on parameters r_a, c_a and instead write r, c to alleviate notation.

To formally define the control problem that we study in this section, let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ denote a complete filtered probability space equipped with a one-dimensional Brownian motion B_t. Fix $b : [0,1] \to \mathbb{R}_+$ a continuous function. The control problem of the player is given by:

$$v_a(x) = \sup_{\alpha \in \mathscr{A}} \mathbb{E}\left[\int_0^\infty re^{-rt}\Big(X_t - \frac{c}{2}\alpha_t^2\Big)dt\right] \qquad \text{s.t. } dX_t = \sqrt{2\big(\alpha_t + b(X_t)\big)}dB_t - dK_t,\ X_0 = x,$$

where X_t, K_t solve the reflection problem i.e $X_t \in [0,1]$.

The following subsection introduces the approach used to solve the control problem: a differential characterization of the problem and the theory of viscosity solutions.

2.1. Differential Characterization: Existence and Uniqueness

The value function of the control problem (and therefore the optimal control) is fully characterized as the solution to a second-order differential equation — the Hamilton–Jacobi–Bellman (HJB) equation — with reflective boundary conditions, which capture the fact that the status quo is reflected on a closed interval.

To state the main result of this subsection, denote the positive part of $y \in \mathbb{R}$ by the subscript $+$, $y_+ := \max\{y, 0\}$, $\mathscr{O}$ the interior of the domain of x, $\mathscr{O} = (0,1)$, and $\partial\mathscr{O}$ the boundary of the domain, $\partial\mathscr{O} = \{0,1\}$, and let $n(x)$ denote the outer normal unit vector, where $n(0) = -1, n(1) = 1$.

Theorem 1. The value function v_a is the unique viscosity solution to the following Hamilton–Jacobi–Bellman equation:

$$rv_a(x) - \sup_{a \in \mathbb{R}_+} \left\{ rx - \frac{rc}{2}a^2 + \big(a + b(x)\big) v_a''(x) \right\} = 0 \quad \text{ on } \mathcal{O} \tag{HJB}$$

with the reflective boundary condition:

$$n(x) v_a'(x) = 0 \quad \text{ on } \partial\mathcal{O}. \tag{BC}$$

Furthermore, v_a is continuous and, whenever v_a'' exists, the optimal control is given by

$$a(x) = \frac{1}{rc} v_a''(x)_+.$$

We will refer to the combination of (HJB) and (BC) as the *reflected problem* (RP) given b.

Note that the value function only solves (RP) in an appropriate weak sense we define below: it is a viscosity solution.[4] Throughout we rely on the use of viscosity solutions, as a number of issues render our problem non-standard and motivates the use of such approach. First, the process degenerates and becomes deterministic if there is no instability. If for some $x \in [0,1]$ $b(x) = 0$, then by setting $a_t = 0$ the player can make the process constant. In particular, this implies the boundary conditions need not be satisfied as players can choose to 'deactivate' the reflection by setting $a(x) = b(x) = 0$ at $x \in \{0,1\}$. Indeed if $b(1) = 0$, it is immediate that player A has no interest to generate instability when the status quo is 1, as A enjoys the maximum possible payoff forever. This effectively makes 1 an absorbing point and the strong boundary condition fails to hold in the usual sense. In general, whether or not the (strong) boundary conditions hold is tightly related to the activity of the other player.

Second, players' best responses generally do not satisfy standard regularity conditions (as Lipschitz continuity), which prevents us from appealing to well-known result for existence and uniqueness. Finally, the value function can be non-differentiable; we will show below it can exhibit a kink in equilibrium. The presence of a kink is more than a technical curiosity and will reflect essential properties of an equilibrium: a kink appears at a stable status quo that is supported but both players threatening to generate enough insta-

[4]For general references on the theory of viscosity solutions of elliptic second order differential equations and its applications to optimal control see Crandall et al. (1992), Fleming and Soner (2006), and Pham (2009).

bility on either side to prevent any deviations. We give more details below when precisely characterizing the value function and equilibrium.

Before we introduce viscosity solutions, observe that the (HJB) equation can be rewritten as:

$$rv_a(x) - rx - b(x)v_a''(x) - \frac{1}{2rc}\left[v_a''(x)_+\right]^2 = 0.$$

For convenience, we define the following notation for the differential operators:

$$\begin{aligned} F_a(x,v,M) &:= rv - rx - b(x)M - \frac{1}{2rc}\left[M_+\right]^2 && \text{for } (x,v,M) \in [0,1]\times[0,1]\times\mathbb{R} \\ B(x,p) &:= n(x)p && \text{for } (x,p) \in \{0,1\}\times\mathbb{R} \end{aligned}$$

that is, (RP) is given by $F_a(x,v_a(x),v_a''(x)) = 0$ on $(0,1)$ and $B(x,v_a'(x)) = 0$ on $\{0,1\}$. We now state the definition of a viscosity solution of (RP):

Definition 2. A function w on $[0,1]$ is a viscosity **subsolution** of (RP) if its upper-semicontinuous envelope w^* is satisfies

$$\begin{aligned} F_a(x_0,w^*(x_0),\varphi''(x_0)) \le 0 &\text{ if } x_0 \in (0,1) \\ \min\{F_a(x_0,w^*(x_0),\varphi''(x_0)), B(x_0,\varphi'(x_0))\} \le 0 &\text{ if } x_0 \in \{0,1\} \end{aligned}$$

for all $\varphi \in \mathscr{C}^2([0,1])$ such that x_0 is a local **maximum** of $w^* - \varphi$.

A function w on $[0,1]$ is a viscosity **supersolution** of (RP) if its lower-semicontinuous envelope w_* is satisfies

$$\begin{aligned} F_a(x_0,w_*(x_0),\varphi''(x_0)) \ge 0 &\text{ if } x_0 \in (0,1) \\ \max\{F_a(x_0,w_*(x_0),\varphi''(x_0)), B(x_0,\varphi'(x_0))\} \ge 0 &\text{ if } x_0 \in \{0,1\} \end{aligned}$$

for all $\varphi \in \mathscr{C}^2([0,1])$ such that x_0 is a local **minimum** of $w_* - \varphi$.

A function w is a **viscosity solution** if it is both a viscosity subsolution and a viscosity supersolution.

Viscosity solutions provide a powerful notion of generalized differentiability which is well adapted to studying HJB-type equations. One canonical intuition to visualize the viscosity approach is to think about fitting smooth test functions — the φ in the definition — equal to the function at a given point but everywhere else above (for a subsolution) or below (for a supersolution) and requiring the differential equation to hold with the appropriate inequality for any such test function.

The proof of Theorem 1 is a combination of two propositions.[5] First, we prove that the value function is a viscosity solution to the stated equation.

Proposition 1 (Optimality)**.** The value function v_a is a viscosity solution to (RP).

Proposition 1 follows from standard dynamic programming arguments and applying Ito's lemma to appropriate test functions, although our setup imposes minimal assumptions.

We then turn to proving we have a unique viscosity solution, therefore corresponding to the value function itself. To do so, we first establish a comparison principle result that will also be of practical interest in characterizing equilibrium properties.

Lemma 1 (Comparison Principle)**.** If $\overline{w}$ is a viscosity supersolution and $\underline{w}$ is a viscosity subsolution to (RP), then $\overline{w} \geq \underline{w}$ in $[0,1]$.

The comparison principle allows us to find bounds for our solution by constructing sub- and supersolutions. Moreover, since existence can be established using general arguments, the comparison principle is instrumental in proving uniqueness.[6]

Proposition 2 (Existence and Uniqueness)**.** There exists a unique viscosity solution to (RP). Furthermore, it is continuous.

The proof of Lemma 1 and Proposition 2 relies on adapting existing techniques from the literature (see Crandall et al. 1992) with arguments that are idiosyncratic to the problem at hand.

2.2. Properties of Best Responses

We now characterize player A's optimal control for an arbitrary strategy by player B. Recall that player A's control is characterized by

$$a(x) = \frac{1}{rc} v_a''(x)_+,$$

[5]Detailed proofs are provided in the Online Appendix.

[6]To prove existence, it is sufficient to exhibit a subsolution (take $\underline{v}(x) := 0$) and a supersolution (take $\overline{v}(x) := 1$) such that the latter is everywhere above the former. We can then construct a solution by taking the pointwise supremum of subsolutions that are everywhere below that supersolution. This is known as Perron's methods in the viscosity solution literature. The comparison principle then immediately implies uniqueness of a viscosity solution.

where the dependence of b is encoded within the value function v_a.

Even if player A's optimal control does depend on b, the following theorem shows that the best response and value function always exhibit a very simple structure characterized by a threshold mechanism.

Theorem 2. (Best Response Characterization) Let b be a continuous function and v_a the solution to problem (RP) given b. The optimal control a^* exists, and the solution to the control problem is fully characterized by two thresholds $\underline{x}_a, \overline{x}_a \in (0,1]$, $\underline{x}_a \leq \overline{x}_a$, such that:

(i) on $[0, \underline{x}_a)$ (*the beneficial instability region*), v_a is strictly convex, and strictly above the identity;

(ii) on $[\underline{x}_a, \overline{x}_a]$ (*the neutral region*), $v_a(x) = x$;

(iii) on $(\overline{x}_a, 1]$ (*the detrimental instability region*), v_a is strictly concave, and strictly below the identity;

(iv) a^* is continuous, strictly positive on $[0, \underline{x}_a)$ and zero elsewhere.

Furthermore, v_a is increasing and twice continuously differentiable everywhere, except possibly at $\overline{x}_a$. If v_a is not differentiable at $\overline{x}_a < 1$, then $\lim_{x \to \overline{x}_a^-} v_a'(x) \geq \lim_{x \to \overline{x}_a^+} v_a'(x)$ (only concave kinks are permissible).

Let us discuss the intuition underlying Theorem 2 and its implications.

First, observe that the threshold structure is a general feature of best responses, regardless of player B's strategy. The optimal strategy for A always consists of generating strictly positive instability when the status quo is unfavorable enough, and doing so in a vanishing manner as the player reaches a 'satisficing' threshold $\underline{x}_a$. The fact that v_a is strictly above the identity in the beneficial instability region captures the idea that A is strictly better off there than if they were able to stay at that status quo forever. Further, the fact that v_a is convex in this region captures the (positive) option value from instability. This option value decreases as player A's share nears $\underline{x}_a$ and the player becomes more prudent as they have more to lose.

The lower threshold $\underline{x}_a$ synthesizes player A's ability to use instability to their advantage and is determined both by b, the discounting factor, and the cost parameter. Essentially, expected gains from instability come from durably experiencing more favorable states. Beyond $\underline{x}_a$, it would be too costly or not beneficial enough to try to generate instability in their favor. This can be because it would require too long a span of instability — entailing too high a cost — to exploit the option value offered by the lower bound and secure durable improvements, or because player B would generate enough instability at more favorable situations for player A so as to prevent them from durably improving their situation there.

Notice that the beneficial instability region $[0, \underline{x}_a)$ is always non-empty: there is always a benefit to generating some instability when the status quo is too disadvantageous. Reflection binds at the lower bound, where the player has the worst possible payoff; for any interior status quo, instability is locally equally likely to make the player worse off or better off. Yet, the fact that there is a worst state generates option value and the strict incentive to increase volatility at the bound spills over and makes it profitable to generate instability in a nearby region. Such a threshold not only always exists for arbitrary b, but it is also always strictly above zero, which demonstrates that instability always enables players to fight off against situations that are too unfavorable.

The upper threshold $\overline{x}_a$ only matters for determining the payoff structure at states in which player A does not contribute to instability. It delineates an intermediate neutral region $[\underline{x}_a, \overline{x}_a]$ where, even though player A chooses not create instability, whatever instability might be generated by the opponent is not harmful (expected payoff are equal to status quo payoffs). For states that are strictly preferred to $\overline{x}_a$ by player A — where they have a lot to lose — whatever instability is generated by the opponent is actively harmful to player A. This is captured by the fact that $v_a(x) < x$ in this region: player A would prefer the status quo to change, and v_a is concave as the option value of instability is negative. Although the neutral region is always non-empty (but possibly consisting of a single point), the detrimental instability region can be empty. Additionally, in general, it need not be the case that $b(x) = 0$ in the neutral region.

The following corollary highlights that the value of b at 1 determines whether or not the upper bound is actively reflecting.

Corollary 1. Fix b. Then, $\overline{x}_a < 1$ if and only if $b(1) > 0$.

Corollary 1 highlights how instability at the extreme states significantly influences the player's behavior in the interior of the domain. If $b(1) = 0$, no amount of instability that player B otherwise generates inside the domain can be harmful to A (the detrimental instability region is empty). As in this case b entails no instability at player A's preferred state ($x = 1$), and as player A would never optimally generate instability at this most favorable status quo, A is then able to make this boundary fully absorbing. What happens at the extremes has drastic consequences everywhere else: whatever instability B otherwise generates is non-harmful and instead benefits player A. On the other hand, if the boundary at 1 is actively reflecting ($b(1) > 0$), player A cannot unilaterally stop the process at 1, and the option value of increasing instability when close to 1 is negative.

Lastly, we provide a monotonicity result.

Proposition 3. If b is non-decreasing on $[0, \underline{x}_a]$, then the optimal control to the problem (RP) is non-increasing.

This intuitively suggests what will be the structure of equilibria: if one player creates more instability when more disadvantaged, but becomes more conservative as the status quo is more favorable to them, the other player will have incentives to do the same.

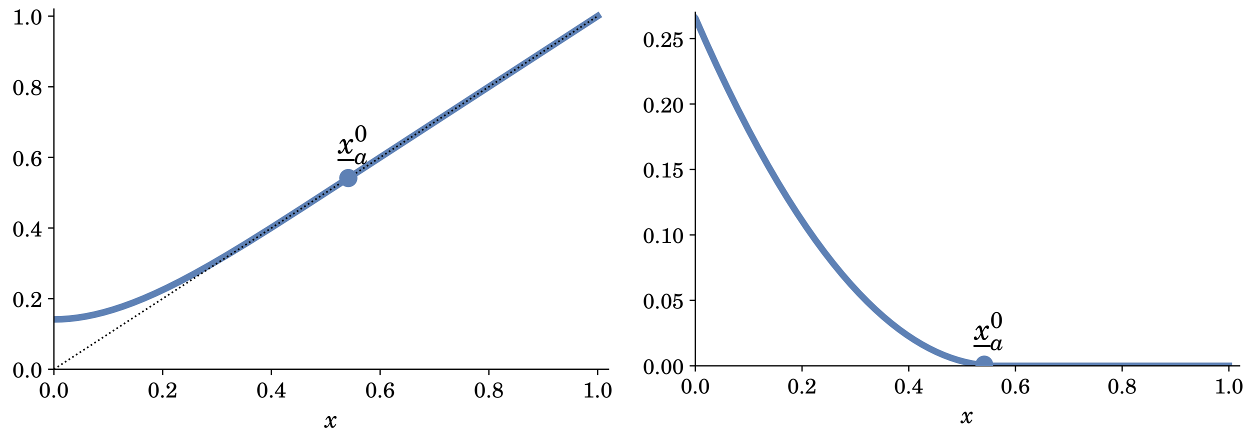
2.3. The Inactive Benchmark

Consider the case in which a player's opponent is fully passive and never generates any instability. We will take player A's viewpoint, with $b(x) \equiv 0$ for all x, so that player A's actions are the only source of instability to the status quo. The analysis of this individual decision-making problem is not only interesting in its own right, but will prove useful both to ground intuition, and to later highlight how equilibrium analysis differs from the the individual problem.

Rewriting the reflected problem explicitly in this case:

$$\begin{aligned} rv_a(x) - rx - \frac{1}{2rc}[v_a''(x)_+]^2 &= 0 \text{ on } (0,1) \\ n(x)v_a'(x) &= 0 \text{ on } \{0,1\} \end{aligned}$$

The HJB equation has a clear interpretation in this case: it relates the instantaneous cost of control at the optimum to the marginal benefit relative to the status quo, which can



(a) **Value Function** (b) **Optimal Strategy**

Figure 1. **Player A in the Inactive Benchmark**

Note: Parameter values are $r_a = 5$ and $c_a = 4$.

seen as the option value of instability. This also highlights why the second-order derivative in this context captures the option value. Indeed, rewrite the HJB as:

$$\underbrace{r(v_a(x) - x)}_{\text{improvement on the status quo}} = \underbrace{\frac{1}{2rc}[v_a''(x)_+]^2}_{\text{option value}} = \underbrace{\frac{rc}{2}a^*(x)^2}_{\text{instantaneous cost of control}}$$

The following proposition refines and strengthens the results of Theorem 2 for arbitrary b to the special case $b \equiv 0$.

Proposition 4 (Properties of the Inactive Benchmark)**.** Let v_a^0 be the value function in (RP) given $b \equiv 0$, and $a^{*,0}$ be the corresponding optimal control. Then, there is $\underline{x}_a^0 \in (0,1]$ such that

(i) on $[0, \underline{x}_a^0)$, v_a^0 is strictly convex, $v_a^0(x) > x$, and $a^{*,0}$ is strictly positive and strictly decreasing;

(ii) on $[\underline{x}_a^0, 1]$, $v_a^0(x) = x$ and $a^{*,0}(x) = 0$.

Moreover, v_a^0 and $a^{*,0}$ are twice-continuously differentiable except possibly at $\underline{x}_a^0 = 1$, with $v_a^{0\prime} \leq 1$.

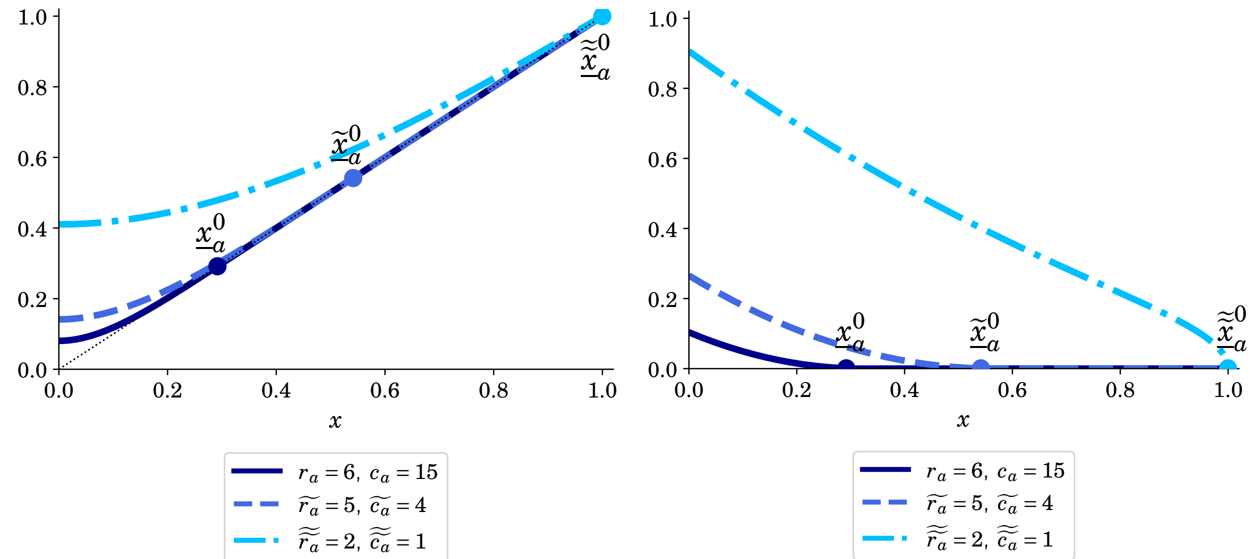
Figure 1 illustrates Proposition 4 with a numerical approximation of the value function and the optimal control of player A in the inactive benchmark for different parameter values. It exhibits the typical best-response structure: the value function is convex and above the identity when x is low enough; it meets the identity at $\underline{x}_a^0$, and remains at the status quo for greater values of x.

It is immediate from Corollary 1 that when $b \equiv 0$ the detrimental instability region is empty, although the intuition is sharper in this special case: If player A fully controls instability, then they will never choose harmful levels of instability as they can always guarantee the status quo. Therefore, they always do at least as well as the status quo, i.e. $v_a^0(x) \geq x$ everywhere.

When the opponent is passive, player A's inaction region $[\underline{x}_a^0, 1]$ determines the states at which, for the given cost and discounting parameters, player A has no possible intrinsic benefit from instability. The active region $[0, \underline{x}_a^0)$ symmetrically delineates the situations where r and c are such that player A can strictly profit from instability. By the same logic, the fact that $a^{*,0}$ is strictly decreasing over the active region $[0, \underline{x}_a^0)$ captures the idea that the return to instability is decreasing as the status quo moves farther away from zero: at more favorable states the improvement on the status quo shrinks and, with it, so does the value to generating instability. Figure 1b below depicts the corresponding optimal control to Figure 1a, illustrating this decreasing behavior. Even without any response from the other player ($b \equiv 0$), structural properties of the problem make it optimal to become monotonically more conservative as the status quo approaches a satisficing point. Notice that the cost and discounting parameters c, r determine not only the position of the satisficing threshold, but also the *speed* at which it is optimal to become more conservative when the status quo is more favorable and the player stands more to lose from instability. This can be intuitively seen from rewriting directly from the HJB, for $x \in [0, \underline{x}_a^0)$:

$$a^{*,0}(x) = \frac{1}{rc} v_a^{0\prime\prime}(x)_+ = \sqrt{\frac{2}{c}(v_a^0(x) - x)}.$$

Hence, player A becomes increasingly more conservative, choosing smaller $a^{*,0}$, as the improvement on the status quo shrinks, and the more so the higher the cost to instability. Of course, this presentation is only heuristic as v_a is implicitly defined — in particular the dependence on the discount rate r is hidden.



(a) **Value Function** (b) **Optimal Strategy**

Figure 2. **Comparative Statics for Player A in the Inactive Benchmark**

We further provide comprehensive comparative statics on r, c. These follow directly from the comparison principle (Lemma 1): it suffices to show that the value function for higher cost/impatience is a viscosity supersolution to the problem for lower cost/impatience. Indeed rewrite the HJB equation as

$$2r^2c(v_a(x) - x) - [v_a''(x)_+]^2 = 0.$$

Assume v_a solves this equation, along with the boundary condition (in the viscosity sense) for r, c, and denote $\underline{x}_a$ the corresponding inaction threshold. Let $\tilde{r}, \tilde{c}$ such that $\tilde{r}^2\tilde{c} \geq r^2 c$. Directly:

$$2\tilde{r}^2\tilde{c}(v_a(x) - x) - [v_a''(x)_+]^2 \geq 0.$$

This inequality is actually an equality on $[\underline{x}_a, 1]$, and if we also have $\tilde{r}^2\tilde{c} > r^2 c$, then this inequality is strict on $[0, \underline{x}_a)$. Moreover, the boundary conditions are still verified in the viscosity sense, which allows us to conclude that v_a is a supersolution in the problem for $\tilde{r}, \tilde{c}$. The next proposition summarizes our comparative statics results:

Proposition 5. Consider two pairs of cost and discounting parameters r, c and $\tilde{r}, \tilde{c}$. Denote v_a^0, $x_a^0, a^{*,0}$ the value function, optimal threshold, and control corresponding to the problem for r, c. Similarly define $\tilde{v}_a^0$, $\tilde{x}_a^0$, $\tilde{a}^{*,0}$, $a^{*,0}$ under $\tilde{r}, \tilde{c}$. If $\tilde{r}^2\tilde{c} > r^2 c$, then,

(i) $\tilde{x}_a^0 \leq x_a^0$, with strict inequality if $\tilde{x}_a^0 < 1$;

(ii) $\tilde{a}^{*,0} \leq a^{*,0}$ on $[0,1]$, with strict inequality on $[0, x_a^0)$; and

(iii) $\tilde{v}_a^0 \leq v_a^0$ on $[0,1]$, with strict inequality on $[0, x_a^0)$.

The interpretation of Proposition 5 is quite natural. For a higher cost/impatience, instability is less profitable overall. As the option value of generating instability is fully due to the player's forward-looking behavior, the more impatient the player is, the lower is the option value provided by the lower bound on the state x. And if impatience affects the marginal benefit to generating instability for a forward-looking player, higher costs raise its marginal cost. As a consequence, higher cost or impatience cause the region where it is beneficial to generate instability to shrink (its upper threshold decreases: $\tilde{x}_a^0 \leq x_a^0$). The instability generated at any given share x is milder ($\tilde{a}^{*,0} \leq a^{*,0}$), and strictly so if the player engages in generating a strictly positive level of instability. This results in lower payoffs $\tilde{v}_a^0 \leq v_a^0$; strictly so when instability is profitable under either pair of parameters.

As a player becomes more patient and faces lower costs to instability, the player may find it worthwhile to generate strictly positive instability everywhere but at 1, which is captured by the fact that $\underline{x}_a^0 = 1$. The player's threshold is also associated with the shape of the player's optimal instability as illustrated in Figure 2b. When player A stops instigating instability at $\underline{x}_a^0 < 1$, then the fact that the flow payoffs are bounded above by 1 (x denotes a share) never comes into play: the upper bound is inactive. In this case, the player's optimal control exhibits a convex shape, and the instability generated by player A vanishes smoothly, with $a^{*,0\prime}(x) \to 0$ as $x \to \underline{x}_a^0$, as observed from the darker solid and dashed lines in Figure 2b. In contrast, we observe that if the discount rate or the cost to instability are low enough, the player adjusts volatility to exactly attain its first best and avoid the upper bound becoming actively reflecting. Then, the player only stops generating instability exactly at $\underline{x}_a^0 = 1$, and we obtain the convex-concave shape for $a^{*,0}$ that we observe in the dashed-dotted line in Figure 2b, associated to instability vanishing abruptly at $\underline{x}_a^0$. We show that this distinction between the cases depicted in Figure 2b is in fact a generic property:

Proposition 6 (Implications of Active Upper Bound). Let v_a^0 be the value function in (RP) given $b \equiv 0$, and $a^{*,0}$ be the corresponding optimal control. Then,

(i) $a^{*,0}$ is convex if and only if $v_{a,-}^{0\prime}(\underline{x}_a^0) = 1$;

(ii) there is $\hat{x}_a \in [0, \underline{x}_a^0)$ such that $a^{*,0}$ is convex on $[0, \hat{x}_a]$ and concave on $[\hat{x}_a, \underline{x}_a^0]$ if and only if $v_{a,-}^{0\prime}(\underline{x}_a^0) < 1$.

Furthermore, $v_{a,-}^{0\prime}(\underline{x}_a^0) = 1$ if $\underline{x}_a^0 < 1$, and $\lim_{x \uparrow \underline{x}_a^0} a^{*,0\prime}(x) = -\infty$ if $v_{a,-}^{0\prime}(\underline{x}_a^0) < 1$.

2.4. Symmetry Properties: the Control Problem for B

Before moving to the characterization of equilibrium, we briefly make explicit symmetric notation and equivalent results for the control problem of player B.

Given a continuous function a, the reflected problem for player B is given by:

$$
\begin{aligned}
r_b v_b(x) - r_b(1-x) - a(x) v_b''(x) - \frac{1}{2 r_b c_b}[v_b''(x)_+]^2 &= 0 \qquad \text{on } (0,1) \\
n(x) v_b'(x) &= 0 \qquad \text{on } \{0,1\}
\end{aligned}
$$

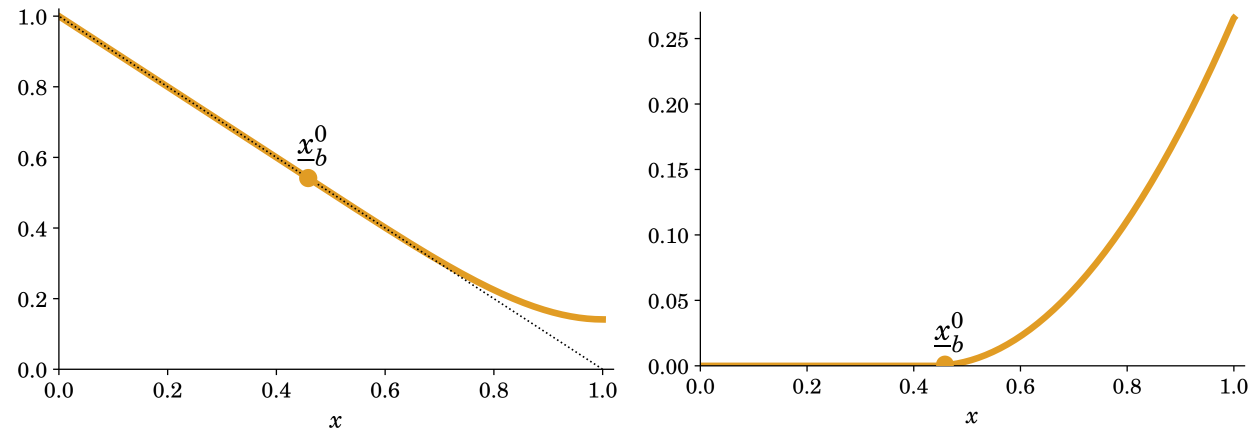
With the change of variable $y = 1 - x$ (inverting the interval by relabeling 1 as 0 and vice versa), the solution to this problem is exactly the solution to player A's problem — up to possibly heterogeneous parameters r_b, c_b. This change of variable amounts to expressing everything in terms of the payoffs to player B instead of player A, which highlights that the problem is symmetric up to a relabeling (and possibly heterogeneous cost and discounting parameters). Therefore, all previous results directly carry symmetrically to player B's problem.

It is important to keep in mind that player B's preferences over the status quo are opposite to player A's. So for instance, whereas we have v_a increasing, v_b is decreasing. Indeed the analogue of Theorem 2 delivers existence of $0 \le \overline{x}_b \le \underline{x}_b < 1$ such that:

(i) $[0, \overline{x}_b]$ is the detrimental instability region for B, i.e. v_b is concave, $v_b(x) \le 1 - x$ and $b^*(x) = 0$;

(ii) $[\overline{x}_b, \underline{x}_b]$ is the neutral region for B, i.e. $v_b(x) = 1 - x$ and $b^*(x) = 0$;

(iii) $(\underline{x}_b, 1]$ is the beneficial instability region for B, i.e. v_b is convex, $v_b(x) > 1 - x$ and $b^*(x) > 0$.



(a) **Value Function** (b) **Optimal Strategy**

Figure 3. **Player B in the Inactive Benchmark**

Note: Parameter values are $r_b = 5$ and $c_b = 4$.

As players' interests are diametrically opposed, beneficial instability regions are located on opposite side of the interval — in each case around the worst status quo for that player. Similarly, player B becomes more conservative as the state converges to its satisficing threshold $\underline{x}_b$ from the right. The value function v_b has the same regularity properties (including the possibility of a concave kink). Figures 3a and 3b show numerical computations for player B's value function and optimal control respectively in the inactive benchmark ($a \equiv 0$).

Lemma 3 has an analogous converse for the same symmetry reasons: if a is non-increasing, then player B's best response to a is non-decreasing. This fully captures the idea that if one player tends to become more conservative when their situation improves, so does the other one.

Properties of the inactive benchmark when $a \equiv 0$ (Propositions 4 and 6) also carry through symmetrically. The value function v_b^0 is convex, decreasing and $v_b^0(x) \geq 1 - x$ everywhere, and it is uniquely characterized by the threshold $\underline{x}_b^0 \in [0, 1)$, where:

(i) on $[0, \underline{x}_b^0]$, $v_b^0(x) = 1 - x$ and $b^{*,0}(x) = 0$;

(ii) on $(\underline{x}_b^0, 1]$, v_b^0 is strictly convex, $v_b^0(x) > 1 - x$, and $b^{*,0}$ is strictly positive and strictly increasing.

Similarly to player A, player B's value function and optimal control are $\mathscr{C}^2$ except possibly at $\underline{x}_b^0 = 0$, with $v_b^{0\prime} \le 1$ (with equality if $\underline{x}_b^0 > 0$). And the optimal control $b^{*,0}$ is convex if and only if $v_{b,+}^{0\prime}(\underline{x}_b) = 1$, and otherwise exhibits a concave-convex structure — being concave on $[\underline{x}_b, \hat{x}_b]$ and convex on $[\hat{x}_b, 1]$, for some $\hat{x}_b \in (\underline{x}_b, 1]$.

Finally, comparative statics from Proposition 5 similarly extend: when r_b or c_b increase, the threshold $\underline{x}_b^0$ does too and instability becomes overall less beneficial for player B. These symmetry properties will be particularly useful in characterizing equilibria, and the relative position of the thresholds $\underline{x}_b^0, \underline{x}_a^0$ will determine the properties of equilibrium.

3. Characterizing Equilibria

We now turn our attention to characterizing (Markov-perfect) equilibria of the game. These will be pairs of strategies (a, b) such that each is a best response to the other, and so any equilibrium will necessarily have to comply with the properties discussed in the previous section. In this section, we first establish three necessary properties inherent to any equilibrium: (i) at most one player is creating instability, (ii) the more favorable the status quo for a player, the lower the instability that player generates, and (iii) each player generates lower instability than they otherwise would were their opponent passive.

This characterization allows us to delineate two possible cases for equilibrium, depending on whether the instability regions in each player's inactive benchmark overlap. If they do not overlap, this gives rise to a unique *accommodating equilibrium*, where players take an accommodating attitude toward the pursuit of a more favorable status quo by their opponents, generating a multiplicity of stable states. If they do overlap, this leads to the existence of *deterrence equilibria*, in which there is a unique stable state quo that is sustained by a deterrence mechanism.

3.1. Properties of Equilibria

We first turn to a crucial property of *equilibria*: the fact that equilibrium instability strategies decouple — at most one player generates instability at any given status quo. Note that this feature is not immediately implied by our characterization of individual best-responses in Section 2.2: there are strategies b for which player A's best response involves generating instability at states x for which $b(x) > 0$. Nevertheless, any equilibrium of the game is uniquely characterized by two thresholds that delineate three regions: a stable region, a region where only player A generates instability, and a region where only player B generates instability. As we show, all equilibria follow this structure. The next proposition summarizes those properties and characterizes the structure of equilibria.

Proposition 7. In any equilibrium, there exist $\underline{x}, \overline{x} \in (0,1)$, $\underline{x} \leq \overline{x}$ such that

(i) $\forall x \in [0, \underline{x})$, $a(x) > 0 = b(x)$;

(ii) $\forall x \in (\overline{x}, 1]$, $a(x) = 0 < b(x)$; and

(iii) $\forall x \in [\underline{x}, \overline{x}]$, $a(x) = 0 = b(x)$.

Furthermore, a (resp. b) is strictly decreasing on $[0, \underline{x})$ (resp. increasing on $(\overline{x}, 1]$), and the equilibrium is uniquely pinned down by $\underline{x}$,$\overline{x}$.

The first thing to note is that Proposition 7 distinguishes between states that trigger instability and those at which stability is attained. The former are those that are deemed excessively unfavorable by either player A — $x \in [0, \underline{x})$ — or player B — $x \in (\overline{x}, 1]$.

The argument for why this 'decoupling' structure of equilibria emerges is simple: owing to the fact they have diametrically opposed interests (constant-sum gross flow payoffs), it is not possible that both players expect to strictly improve on the same status quo. At most one of the players sees an advantage to generating instability at any given status quo. We know that at extremes states, $x = 0$ and $x = 1$, the disadvantaged player will actively push back by creating instability. After all, they have nothing to lose and, while costly, instability can only improve their situation. Then, due to the fact that *for any* strategy of their opponent the set of states at which they find it profitable to generate instability is convex and includes their most unfavorable state (as shown in Theorem 2), we obtain the existence of these three regions.

Second, Proposition 7 tells us as players benefit from a larger share of the available benefits, they become more conservative in how much volatility they create. Recall that the benefit to instability derives solely from the option value provided by the finiteness of resources being shared, as the is no immediate gain to instability when $x \in (0,1)$. However, as there is a natural lower bound on how unsatisfactory the outcome can be, patient players may want to take a calculated risk to reap the benefits of this option value. The proof follows from the fact that equilibrium strategies exhibit this decoupled structure, combined with the fact that if the opponent is unresponsive to instability, the optimal control is monotone in the state just as in the inaction benchmark discussed in Proposition 4.

The next property of equilibria relates the players' equilibrium strategies with their optimal instability strategy in the inactive benchmark case. Let us start by introducing some notation. Denote player A's value function given an arbitrary continuous b by

$$v_a(x \mid b) := \max_{\alpha \in \mathscr{A}} \mathbb{E}\left[\int_0^\infty e^{-r_a t} r_a \left(X_t - c_a \frac{\alpha_t^2}{2}\right) d_t\right] \qquad \text{s.t. } dX_t = \sqrt{2(\alpha_t + b(X_t))} dB_t - dK_t,\ X_0 = x,$$

and the associated optimal control by $\alpha^*(x \mid b)$.

While it is tempting to think that the player always attains the highest expected payoff when their opponent is passive ($b \equiv 0$), this is not the case. Player B could potentially take A's stead in generating optimal instability and saving A the cost of doing so, that is

$$v_a(\cdot \mid \alpha^{*,0}) \geq v_a^0$$

with the inequality being strict for small enough x. However, in equilibrium, it is indeed true that player A cannot be better-off than if facing a passive opponent:

Proposition 8. In any equilibrium, $v_a \leq v_a^0$ and $v_b \leq v_b^0$.

The result derives from two observations. First, that v_a is a subsolution to (RP) in the inactive benchmark case, for which v_a^0 is a solution. Second, from the fact that, from the comparison principle (Lemma 1), we know that any subsolution is weakly smaller than a supersolution — and thus, than a solution.

The benchmark describing what the player could and would do if their opponent were to play passively determines the structure of equilibrium. As previously stated, in general there exist strategies b such that player A's best response will exhibit a threshold above $\underline{x}_a^0$ (i.e. player A's instability region would extend over the one in the inactive benchmark). In a sense, such a b would have to have a structure that benefits A above $\underline{x}_a^0$. However, *in equilibrium* the optimal strategy b of the opponent will never be beneficial to player A because they have diametrically opposed interests: it is not possible that both players simultaneously benefit from instability in equilibrium. This is a direct implication of Proposition 8: $a^{*,0} = 0 \iff v_a^0(x) \leq x \implies v_a(x) \leq x \iff a^*(x) = 0$, and symmetrically for player B. In other words, the intuition that if instability is not beneficial at a given point when $b \equiv 0$, it is still not beneficial when $b \neq 0$ is *true in equilibrium* but only in equilibrium. Additionally, the optimal strategy of the inactive benchmark and the inaction threshold $\underline{x}_a^0$ in particular can be interpreted as the players' ability to threaten their opponent. This underpins the argument that the inactive case discussed in Section 2.3 is indeed the right benchmark. Consequently, we obtain the immediate corollary:

Corollary 2. In any equilibrium, $\underline{x} \leq \underline{x}_a^0$ and $\underline{x}_b^0 \leq \overline{x}$.

While Propositions 7 and 8 deliver necessary properties of any equilibrium, they are silent about the existence of equilibria. The remainder of this section is devoted not only to showing their existence, but also to further specializing the characterization of equilibria by delineating the two possible kinds of equilibrium, which depend on parameter values.

3.2. Accommodating Equilibria

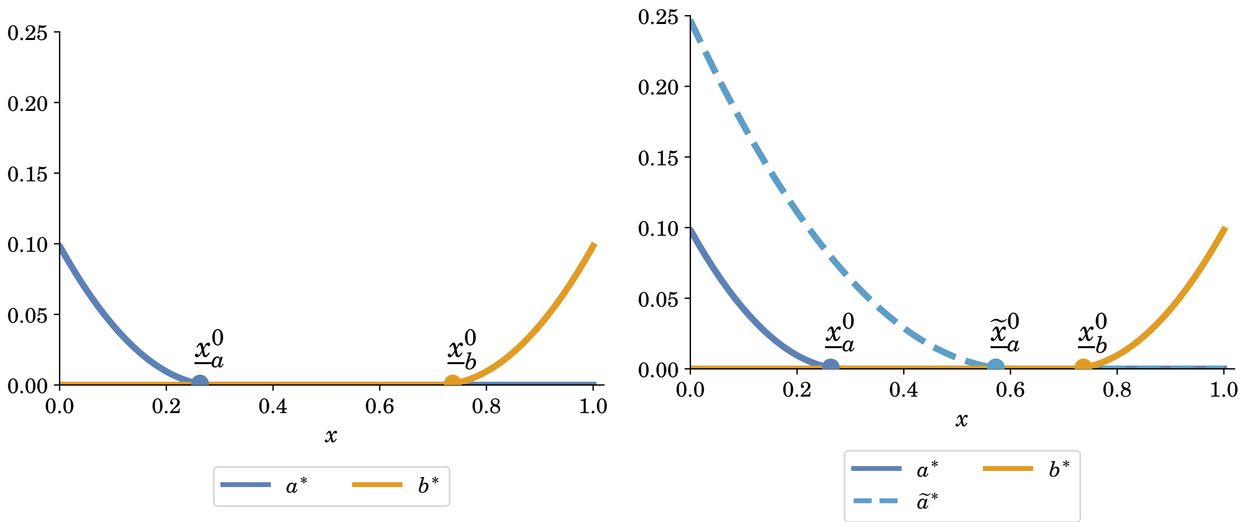
In this section we show that, under some conditions, the unique equilibrium is such that players behave as if their opponent is passive. In other words, they behave as if they were individually controlling volatility.

The main result in this section is as follows:

Theorem 3. If $\underline{x}_a^0 \leq \underline{x}_b^0$, there is a unique equilibrium given by $(a^*, b^*) = (a^{*,0}, b^{*,0})$. Moreover, at any equilibrium such that $\underline{x} < \overline{x}$ it must be the case that $(\underline{x}, \overline{x}) = (\underline{x}_a^0, \underline{x}_b^0)$.

Whenever there is a status quo such that neither of players wants to increase instability even if their opponent were passive, then equilibrium behavior is everywhere as if their opponent were indeed passive. In short, equilibrium behavior is accommodating towards their opponent's aspirations to obtain a better outcome for themselves by creating some instability; players never 'push back' against one another. The comparison principle also allows direct comparison of equilibrium payoffs. In an accommodating equilibrium, each player is just as well-off as in the inactive benchmark in the region where they are the one generating instability, but strictly worse-off in the region where the other player is generating instability.

It is worth emphasizing that an accommodating equilibrium, when it exists, must be the unique equilibrium. In other words, when the inactive benchmark is such that there



(a) (b)

Figure 4. **Equilibrium Strategies in an Accommodating Equilibrium**

Note: Parameter values are $r_a = r_b = 7$ and $c_a = c_b = 15$ for the solid lines, and $\widetilde{r_a} = 4$, $\widetilde{c_a} = 5$ for the dashed line.

is no status quo where both players would be willing to generate instability if the other were inactive, then this is the only equilibrium outcome. Given that, from Proposition 5, $\underline{x}_a^0$ (resp. $\underline{x}_b^0$) is decreasing (resp. increasing) with respect to r_a and c_a (resp. r_b and c_b), equilibrium behavior will be accommodating if and only if players' impatience enough and costs to generating instability are high enough.

Another noteworthy feature of accommodating equilibria is that they exhibit an interval $[\underline{x}_a^0, \underline{x}_b^0]$ of stable states where the status quo prevails. This interval can be very large, as in Figure 4a where approximately every state between 1/4 and 3/4 is stable. There, the inability to profitably generate instability means that both players are willing to accept a large range of states. As a consequence, states that can be potentially much more strongly preferred by one player than another can be sustained in the long run.

Proposition 5 also immediately implies a comparative statics result for accommodating equilibria with respect to players' impatience and cost parameters. If a player's impatience or cost increases, then the set of equilibrium stable states expands and includes states that are strictly less favorable to that player; and vice-versa. In Figure 4b, we depict how the equilibrium strategies change when player A becomes more patient and has a lower cost to instability. The resulting instability region for player A is larger, and the stable region shrinks: player A will now find it worthwhile to generate instability at states that were previously stable.

3.3. Deterrence Equilibrium

What if both players have a low enough cost to generating instability, or are patient enough, such that there exists a region where both players would like to generate instability if the other one were inactive? In other words, is there an equilibrium when $\underline{x}_a^0 > \underline{x}_b^0$? From previous result, it is immediate that there cannot be an accommodating equilibrium in that case: both player using their inactive benchmark strategy would lead to both generating strictly positive instability at some status quo, contradicting Proposition 7.

As we have seen from the comparative statics analysis for accommodating equilibria, as players are more patient and generating instability is less costly, the stable region shrinks. And, for low enough costs and impatience parameters, we eventually obtain $\underline{x}_b^0 < \underline{x}_a^0$. Our first observation is that if, $\underline{x}_b^0 < \underline{x}_a^0$, in any equilibrium, it must be the case that the stable region is reduced to a single point.

Lemma 2. If $\underline{x}_a^0 > \underline{x}_b^0$, then at any equilibrium $\underline{x} = \overline{x}$.

To see why this must be the case, note that if $\underline{x} < \overline{x}$, both players' value functions must equal the identity on $[\underline{x}, \overline{x}]$ as on that region no one is generating instability. From Corollary 2, it must be that at least one of the players would like to instigate instability were their opponent passive throughout, i.e. $\underline{x} < \underline{x}_a^0$ or $\underline{x}_b^0 < \overline{x}$. Lemma 2 then shows that if a player generates instability at a given status quo when their opponent is passive throughout (as in the inactive benchmark), then they would do the same in any equilibrium in which their opponent is passive around this state. Moreover, note that Lemma 2 and Corollary 2 combined imply that if $\underline{x}_a^0 > \underline{x}_b^0$, then $\overline{x} = \underline{x} \in [\underline{x}_b^0, \underline{x}_a^0]$.

Our second observation is that, at any equilibrium such that $\underline{x}_a^0 > \overline{x} > \underline{x}_b^0$, the players' equilibrium strategies are vanishing abruptly at $\overline{x}$. To see this, note that the player A's value function is a viscosity solution v_a that satisfies $F_a(x, v_a(x), v_a''(x)) = 0$ on $[0, \overline{x}]$, $B(0, v_a'(x)) = 0$, and $v_a(\overline{x}) = \overline{x}$. Consequently, we will have that the left-derivative of the value function at $\overline{x}$ is strictly smaller than one, as $v_a \leq v_a^0$ and $v_{a,-}'(\overline{x}) \leq v_{a,-}^{0\prime}(\overline{x}) < v_{a,-}^{0\prime}(\underline{x}_a^0) \leq 1$, where the last inequality follows from the fact that v_a^0 is strictly convex on $(\overline{x}, \underline{x}_a^0)$. As, owing to the regularity of our problem, we can derive, for $x \in (0, \overline{x})$,

$$a^{*\prime}(x) \propto v_a'''(x) = r^2 c \frac{v_a'(x) - 1}{v_a''(x)} < 0,$$

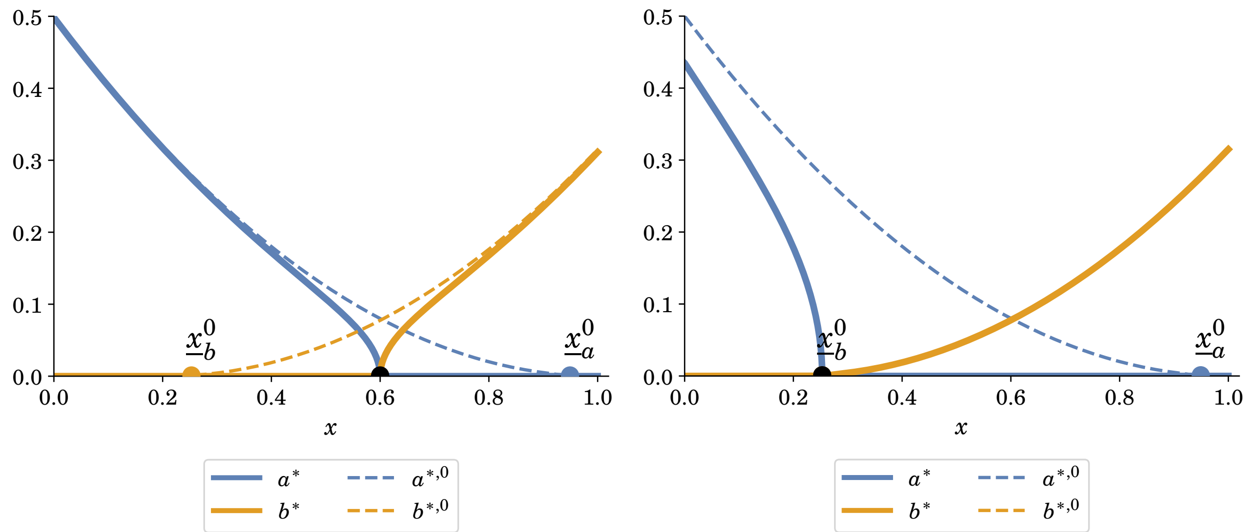
we find that $a^{*\prime}(x) \to -\infty$ as $x \uparrow \overline{x}$, given that the numerator is bounded away from zero and strictly negative $v'_{a,-}(\overline{x}) < 1$, and the denominator is vanishing.

We highlight that this second observation suggests that at any equilibrium such that $\underline{x}_a^0 > \overline{x} > \underline{x}_b^0$, players behave as if $\overline{x}$ is an actively reflecting boundary. Indeed, the fact that player A's equilibrium strategy vanishes abruptly at $\overline{x}$ is reminiscent of how the optimal control in the inactive benchmark case is affected by an actively reflecting upper bound (Proposition 6): if $v^{0\prime}_{a,-}(\underline{x}_a^0) < 1$, then $a_-^{*,0\prime}(\underline{x}_a^0) = -\infty$.

Combined, both these observations suggest a constructive method to characterize any equilibrium: take an status quo $\overline{x} \in (\underline{x}_b^0, \underline{x}_a^0)$ and solve for the player A's (resp. B's) unique viscosity solution to the inactive benchmark problem on $[0, \overline{x}]$ (resp. $[\overline{x}, 1]$), as if reflection occurred at $\overline{x}$ instead of at 1 (resp. at 0). Then, solve the HJB on the region on which the player is inactive taking the opponent's strategy as given with the appropriate boundary conditions, piece the two together, and verify that the resulting function is a viscosity solution to the original problem taking the opponent's strategy as given. In particular, the resulting function needs not only to be continuous at the threshold, but it also cannot exhibit a convex kink at $\overline{x}$. The observations above indicate that any equilibrium must conform with this construction, and thus it is pinned-down by the threshold $\overline{x}$. The question of existence of an equilibrium can then be rephrased as follows: is there an state $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0]$ for which such a construction holds? The next result answers this question affirmatively and provides an exhaustive characterization of the set of equilibria.

Theorem 4. Suppose that $\underline{x}_a^0 > \underline{x}_b^0$. A pair of strategies (a^*, b^*) is an equilibrium if and only if $a^*(x) = \mathbf{1}_{(x<\overline{x})} \frac{1}{r_a c_a} v_a''(x)$ and $b^*(x) = \mathbf{1}_{(x>\overline{x})} \frac{1}{r_b c_b} v_b''(x)$, where v_a and v_b are the unique viscosity solutions to the respective inactive benchmark problems on $[0, \overline{x}]$ and $[\overline{x}, 1]$, and $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0, 1)$.

The proof verifies that any equilibrium needs to satisfy the construction laid out above, and that such a construction is successful in characterizing equilibrium viscosity solutions



(a) **Equilibrium with $\underline{x} = \overline{x} = .6$**

(b) **Equilibrium with $\underline{x} = \overline{x} = \underline{x}_b^0$**

Figure 5. **Equilibrium Strategies in a Deterrence Equilibrium**

Note: Parameter values are $r_a = 5$, $c_a = 6$, $r_b = 6$, and $c_b = 15$.

whenever $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0,1)$. Note that $\overline{x}$ can never be exactly equal to 0 or 1 because, as proved in Theorem 2, $v_a(0), v_b(1) > 0$: at extreme states, the player with nothing to lose will generate strictly positive instability in equilibrium.

At an equilibrium, stability is sustained via deterrence: if their opponent were to not react, both players would like to destabilize the status quo in hope for an improvement of their situation at $\overline{x}$. It is exactly because opponents would react and push back, and would do so with enough intensity that $\overline{x}$ is indeed a stable status quo. When players are too impatient and face too high costs to instability we obtain a unique accommodating equilibrium with a thick region of stable states, $[\underline{x}_a^0, \underline{x}_b^0]$. The situation reverses when they are patient and costs are low: we obtain multiple equilibria, each of which pinned-down by the unique stable state they imply $\overline{x} = \underline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0,1)$.

The equilibrium strategies in accommodating and deterrence equilibria also exhibit meaningfully different properties: player A's (B's) equilibrium strategy is convex if $\underline{x}_a^0 \leq \underline{x}$ ($\overline{x} \leq \underline{x}_b^0$) and convex-concave otherwise. This can be shown directly from Property 10. When parameters induce an equilibrium of accommodating type as in Figure 4a, both players' equilibrium strategies are convex and instability vanish smoothly as they are in this sense unconstrained (thresholds are not enforced by the other player's strategy but by each player's own cost and impatience). By contrast, when the parameters are such that equilibrium are of deterrence type and the unique stable status quo lies strictly between $\underline{x}_b^0$ and $\underline{x}_a^0$ — as in Figure 5a — equilibrium strategies are convex-concave and have infinite slopes at the stable status quo, just as they do in the inactive benchmark when the upper bound becomes a binding constraint. This again captures the constrained nature of a deterrence equilibrium: it is as if the other player is acting as a reflecting barrier at the stable status quo, and each player is solving their own problem on a truncated domain. In Figure 5b, we exhibit the case of a second deterrence equilibrium in which the unique stable status quo coincides with $\underline{x}_b^0$. In such case, player B's equilibrium strategy also coincides with their optimal control in the inactive benchmark. With a reflecting boundary at $\overline{x} = \underline{x}_b^0$ and player B's optimal control would not be affected — explaining the convex structure of the control — whereas it curtails player A's ambitions of reaching more favorable states ($\underline{x}_a^0 > \overline{x} = \underline{x}_b^0$), giving rise to the convex-concave structure of their equilibrium instability strategy.

We briefly remark on the existence and interpretation of a concave kink in the value function. There is a kink only in one very specific situation: when there is a single stable status quo and the threats of instability on both sides are high enough to enforce it. This is precisely the intuition behind a deterrence equilibrium. Indeed, if there is a kink at $\overline{x}$, then b is strictly positive on $(\overline{x}, 1]$ but zero at $\overline{x}$, which implies that $\overline{x}$ is an status quo at which neither player generates instability. If there were a convex kink at $\overline{x}$, v_a would be increasing faster to the right of $\overline{x}$ than to the left, making it profitable for player A to strictly increase volatility in a way that pushes the process up and symmetrically for player B. The fact that only concave kinks are possible can be interpreted as each player fighting back 'hard enough' towards the stable status quo, so as to dissuade the other player from attempting to further improve their situation. In a loose sense, it is exactly the fact that player B pushes back by abruptly increasing instability to the right of the stable status quo that renders it absorbing by making the slope of v_a become discontinuously flatter. This deters player A from taking action as it would be too costly to push the process beyond such a point, and again a symmetric argument holds for player B.

To conclude this section, we note that the above observations highlight once again that these thresholds capture the maximal threat power of each player: there is no equilibrium with a stable status quo more favorable to player A (resp. B) than $\max\{\underline{x}_a^0, \underline{x}_b^0\}$ (resp. $\min\{\underline{x}_a^0, \underline{x}_b^0\}$) and less favorable than $\min\{\underline{x}_a^0, \underline{x}_b^0\}$ (resp. $\max\{\underline{x}_a^0, \underline{x}_b^0\}$). This then reduces comparative statics on the set of equilibria to comparative statics on $\underline{x}_a^0$ and $\underline{x}_b^0$ as given by Proposition 5: If player A becomes more patient or faces lower costs to instability, then the set of stable states as given by any equilibrium ($[\min\{\underline{x}_b^0, \underline{x}_a^0\}, \max\{\underline{x}_b^0, \underline{x}_a^0\}] \cap (0,1)$) becomes more favorable to the player, increasing in the strong set order; and a symmetric argument holds for player B.

Consider for illustration an accommodating equilibria and assume that $X_0 = 0$ i.e we start in A's instability region (at A's worst possible payoff). Lowering costs to instability or impatience of player A will extend its instability region i.e. push the threshold $\underline{x} = \underline{x}_a^0$ towards the right. For small enough changes, B's equilibrium strategy does not change as we remain in a accommodating equilibrium. This is pointwise welfare improving for A in its instability region $[0, \underline{x}_a^0]$ by the comparison principle. When the process hits the point where $\underline{x}_a^0 = \underline{x}_b^0$, this is the limit case where accommodating and deterrence equilibrium are confounded. If we keep lowering r_a, c_a, we now move into deterrence equilibria. This has ambiguous effects on equilibrium because of potential multiplicity, but there always exists an equilibrium that is an improvement for A when we decrease r_a, c_a — strictly so if $\underline{x}_a^0$ remains strictly below 1.

4. Equilibrium Dynamics

What is the effect of players using instability on the dynamics of social change? How do equilibrium dynamics of social instability translate into the evolution of the status quo? The precise characterization of equilibrium in the previous section can be used to answer these questions directly.

A salient characteristic of our model is that all equilibria (of accommodating or deterrence type) display a form of path dependency. Consider an arbitrary equilibrium with thresholds $\underline{x}, \overline{x}$ partitioning the state space $[0,1]$, and denote X_0 the initial point of the process. It is clear that if we start at a stable state, $X_0 \in [\underline{x}, \overline{x}]$, this will remain the status quo for ever. This is by definition of stable states since no player generates any instability.

Moreover, if the process starts in say A's instability region $[0, \underline{x}]$, it will also remain in this region — and similarly for B's instability region $[\overline{x}, 1]$. This comes directly from continuity of the process and since the outer boundary (0 or 1 respectively) is reflecting and the inner boundary ($\underline{x}$ or $\overline{x}$ respectively) is absorbing. Therefore, if the process starts from a given player's instability region it can at most hit its inner boundary and remain there ever after, but it cannot cross over to the other player's instability region. This is a striking feature as it means that whichever player is most disadvantaged at the initial time will remain so forever in equilibrium, and can at most hope to reach the status quo defined by the inner boundary of its equilibrium instability region.

Does the process converge in the long run towards a stable status quo? Or does instability perpetuate if we start in an instability region? Given the previous discussion, it seems intuitive that if there is (probabilistic) convergence from an instability region, it will be towards its inner boundary — the next proposition confirms this intuition.

Proposition 9. Let X_t be the process associated to equilibrium strategies a^*, b^*, and denote $\underline{x}, \overline{x}$ the corresponding equilibrium thresholds. Then, (i) if $X_0 \in [\underline{x}, \overline{x}]$, $X_t = X_0$ for all t; and (ii) if $X_0 < \underline{x}$ (resp. $> \overline{x}$), X_t converges almost surely to $\underline{x}$ (resp. $\overline{x}$).

For the case $X_0 \in [\underline{x}, \overline{x}]$, the proof of Proposition 9 is trivial given that the process is degenerate and there is no instability. For $X_0 < \underline{x}$, we can use a constructive approach to show that X_t is a submartingale. Indeed, construct the process Y_t defined by $dY_t := \sqrt{2a^*(|X_t|)}dB_t$. Because of the structure of a^*, Y_t has absorbing boundaries on $[-\underline{x}, \underline{x}]$, and therefore we can verify that it is a martingale using boundedness of a^* and the optional stopping theorem. By using pathwise uniqueness of the solution X_t, we can argue that $Y_t = |X_t|$, that is, Y_t is the mirror image of X_t without the reflection at 0 — this is done by fixing a Brownian path $B_t(\omega)$, which uniquely pins down $X_t(\omega)$ by pathwise uniqueness. Then, we argue that Y_t and X_t can only cross the origin at the same time and must be either identical or mirrored between two hitting times of zero (since they have the same increments). Since the absolute value is a convex function, we conclude that X_t is a submartingale, and by the martingale convergence theorem it must converge almost surely. We can then prove that X_t converges almost surely to $\underline{x}$ by contradiction, since convergence to any $x < \underline{x}$ would only be sustainable under a measure zero trajectory for the Brownian motion. The argument for $X_0 > \overline{x}$ is symmetric using a similar construction around 1.

Proposition 9 entails that instability is decreasing in the long run. As players approach a stable status quo, whichever player is generating instability becomes more conservative — this is a consequence of the properties of best response which carries through in equilibrium. Therefore, in the long run stability prevails.

5. Discussion

We now discuss a number of variations on our model.

Exogenous instability. To clearly identify the strategic incentives to generate instability, we focused on the case in which any instability is endogenous. Given that our best-response characterization allows for arbitrary strategies by the opponent, and as these correspond to continuous exogenous state-contingent volatility structures, all in subsections 2.1 and 2.2 holds identically when allowing for exogenous instability sources (independent from α_t and β_t, conditional on X_t). Considering a fixed exogenous level of instability $\sigma > 0$, such that $dX_t = \sqrt{\alpha_t + \beta_t + \sigma^2} dB_t - dK_t$, an equilibrium exists.[7] However, while at any equilibrium there is a unique state $\bar{x} \in (0,1)$ at which $a^*(\bar{x}) = b^*(\bar{x}) = 0$, the modified model (mechanically) exhibits perpetual instability.

Costs to instability. While we relied on quadratic costs for expositional convenience, provided enough regularity,[8] results generalize to smooth, strictly convex costs to instability. In particular, the proofs for the threshold structure of best-responses (and other properties in Theorem 2), monotonicity, and equilibrium characterization can be adjusted to accommodate general cost structures. The HJB equation would be given by

$$rv_a(x) - \sup_{a \in \mathbb{R}_+} \left\{ rx - rc(a) + (a + b(x)) v_a''(x) \right\}.$$

Moreover, as $rc'(a^*(x)) = v_a''(x)_+$, whenever $a^*(x) > 0$ we would then have

$$r(v_a(x) - x) - b(x) rc'(a^*(x)) = rc'(a^*(x)) a^*(x) - rc(a^*(x)),$$

[7]It is easy to show that the unique viscosity solution v_a given an arbitrary continuous b is now thrice-continuously differentiable, and that the first three derivatives are bounded. Existence of an equilibrium then follows by an application of Arzelà–Ascoli theorem and Schauder's fixed point theorem.

[8]In particular, costs need to be sufficiently smooth, strictly increasing and strictly convex on $\mathbb{R}_+$, with $0 = c(0) = c'(0)$. Although it goes beyond the scope of this paper to characterize its limits, the proof strategy to (a version of) Theorem 1 extends given enough regularity on the cost function.

from which one can obtain that monotonicity of b implies monotonicity of a^*.[9]

State space. A substantive assumption in our model is that the state space lies on a compact interval, as it is the option value provided by the lower bound that induces players to generate strictly positive instability. Absent a lower bound on the state space, players would have no desire to generate instability unless they were not risk-neutral. A similarly conclusion would hold if the boundaries were absorbing rather that reflecting.

Terminal payoff. Finally, we consider the case of having a terminal payoff, whereby the instead of accruing a flow benefit, players accrue that payoff only when the both players generate no instability. This can be seen as an extreme form of conflict, as creating instability fully deprives the opponent of any flow benefit. Immediately, for a given strategy of the opponent b, one can see that player A's optimal control would need to satisfy $a(x) = 0$ whenever $b(y) > 0$ for any $y \geq x$, and thus we would have decoupling for any best response. Heuristically, in an inactive benchmark ($b \equiv 0$) one would expect player A's value function to solve

$$rv_a(x) = \max\left\{rx, \frac{1}{2rc}[v_a''(x)_+]^2\right\} \qquad \text{on } (0,1)$$

under boundary conditions $v_a'(0) = 0, \quad v_a(1) = 1$, with the control being given by $a^{*,0}(x) = \frac{1}{rc}v_a''(x)_+$. Differently from our model, we note that in such case the instability would be increasing rather than decreasing in the inactive benchmark. Such a result is reminiscent of Moscarini and Smith's (2001), but where, instead of learning, one would have pure instability. Focusing on monotone strategies, a construction of accommodating equilibria with a region of stable states given by $[x_a^0, x_b^0]$ (if $x_a^0 < x_b^0$), and of deterrence equilibria with a unique stable state $\overline{x} \in [x_b^0, x_a^0]$ (if otherwise) would be immediate,[10] with instability greatest at states nearing the region of stable states.

Our model's novel approach to the mechanics of instability and its strategic importance in situations of social instability opens several paths for future investigation. It demonstrates that the possible endogeneity of instability generates non-trivial dynamics

[9] We thank Yu Fu Wong for having pointed out that monotonicity would extend for general cost structures in the individual decision-making case — corresponding to our inactive benchmark with $b \equiv 0$.

[10] Note that at a deterrence equilibrium, no player has an incentive to generate instability at $\overline{x}$ as this would lead to permanent instability and thus a null benefit to instability to both players.

that should be further investigated, notably to better understand the interaction of various conflict mechanisms in richer environments and their applications to concrete situations of conflict, bargaining, and related settings.

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Appendix A. Preliminaries

A.1. Stochastic Differential Equations with Reflection

Consider our equation of interest, for a given a, b continuous measurable functions:

$$dX_t = \sqrt{2\big(a(X_t) + b(X_t)\big)}dB_t - dK_t$$

With say that X_t, K_t solve the reflection problem on $\mathcal{O} := (0,1)$ if they are the continuous $\mathcal{F}_t$-adapted processes such that, denoting by $|K|_t$ the total variation of the process K_t such that (i) $dX_t = \sqrt{2\big(a_t + b(X_t)\big)}dB_t - dK_t$, (ii) $X_t \in [0,1]$ a.s., and (iii) K_t is non-decreasing, $|K|_t = \int_0^t \mathbb{1}_{X_t \in \{0,1\}} d|K|_s$, and $K_t = \int_0^t n(X_s) d|K|_s$, where $n(\cdot)$ denotes the unit outward normal vector to $\mathcal{O}$, that is, $n(1) = 1$, $n(0) = -1$.

K_t is the local time of the process at the boundary — it minimally pushes X_t back inside of the domain (towards the inner normal) if it hits the boundary by compensating the variations that would make X_t exit the domain. Lions and Sznitman (1984) show that such processes are uniquely defined in much more general reflecting domains, essentially under assumptions guaranteeing that the stochastic differential equation (SDE) without reflection has a strongly (pathwise) unique solution.[11]

A.2. Test functions and Second-order Semijets

We recall the definition of second-order semijets. The second order subjet of v at $x_0 \in (0,1)$ is denote by $J^{2,-}_{[0,1]}v(x_0) \subset \mathbb{R}^2$ and defined as:

$$(p,M) \in J^{2,-}_{[0,1]}v(x_0) \Longleftrightarrow v(x) \geq v(x_0) + p(x-x_0) + \frac{1}{2}M(x-x_0)^2 + o(|x-x_0|^2) \quad \text{as } x \to x_0$$

Because the bounds play a special role, when $x_0 \in \{0,1\}$ obviously x can only converge to x_0 from one side. In general, we need to consider the closure of the subjet $\overline{J}^{2,-}_{[0,1]}v(x)$ in the definitions to properly define the viscosity characterization (at the boundary and points of non-differentiability) — see Crandall et al. (1992) for more details.

To relate the subjet with our definition of viscosity solutions in terms of test functions, we recall a classical result. $(p,M) \in J^{2,-}_{[0,1]}v(x_0)$ if and only if there exists a $\mathscr{C}^2$ function ϕ such that x_0 is a local maximum of $v - \phi$ and $\phi'(x_0) = p$, $\phi''(x_0) = M$. It is without loss to require the maximum to be global and to impose $\phi(x_0) = v(x_0)$. In other words, the subjet contains the first and second order derivative values that are admissible for a smooth function ϕ that lies everywhere strictly *below* v (hence the *sub*jet term) and equals v at x_0. This captures

[11] In general, this is not directly applicable to our equation — it is well known since the seminal paper of Yamada and Watanabe (1971) that pathwise uniqueness of solutions to SDEs of the form $dX_t = \sigma(X_t)dB_t$ is difficult to guarantee beyond the general condition that σ is Hölder continuous with coefficient at least 1/2. This condition clearly does not hold for general a, b continuous in our model. However, subsequent work has improved on the Hölder-1/2 condition for specific cases. For our case, the presence of the reflection helps guarantee existence and pathwise uniqueness although it might actually not hold for the unbounded domain. In particular, Bass and Chen (2005) proved that under mild regularity condition, the one-sided reflection problem has a pathwise unique for a α-Hölder diffusion coefficient, $\alpha \in (0,1/2)$. Bass et al. (2007) extends and provides a different proof of the result. Their proof strategy for the one-sided reflection essentially covers our case of interest and easily extends to having a second reflecting barrier: this guarantees existence pathwise uniqueness of a solution to our equation with one-sided reflection at zero, we can then complete the proof by using the analytical apparatus of Lions and Sznitman (1984) or the original approach by Skorokhod (1961) to prove existence and pathwise uniqueness with the second reflecting barrier given pathwise uniqueness of the one-sided reflecting process.

all the relevant differential information on v and can indeed be interpreted as a notion of differentiability for non-differentiable functions. The superjet is defined symmetrically, but considering a convex quadratic approximation (or a smooth test function) from above. We denote it by $J^{2,+}_{[0,1]}v(x_0) \subset \mathbb{R}^2$ and it is defined as:

$$(p,M) \in J^{2,+}_{[0,1]}v(x_0) \Longleftrightarrow v(x) \leq v(x_0) + p(x-x_0) + \frac{1}{2}M(x-x_0)^2 + o((x-x_0)^2) \quad \text{as } x \to x_0$$

Similarly we denote by $\overline{J}^{2,+}_{[0,1]}v(x_0)$ the closure of the superjet. The analogue result holds for test functions: $(p,M) \in \overline{J}^{2,+}_{[0,1]}v(x_0)$ if and only if there exists a $\mathscr{C}^2$ function ϕ such that x_0 is a local (wlog global) minimum of $v-\phi$ and $\phi'(x_0) = p$, $\phi''(x_0) = M$ (wlog $\phi(x_0) = v(x_0)$).

In the following proofs, we alternate between the (equivalent) formulation of viscosity properties in terms of test functions and semijets, in order to choose the most convenient and intuitive approach.

Appendix B. Omitted Proofs

The proof of Theorem 1 (Propositions 1 and 2) is rather technical and cumbersome and can be found in the online appendix.

B.1. Proof of Theorem 2 (Best-Response Characterization)

The proof of Theorem 2 consists of the following intermediary results:

Proposition 10. There exists $\underline{x}_a, \overline{x}_a \in (0,1]$, $\underline{x}_a \leq \overline{x}_a$, such that: (i) on $[0,\underline{x}_a)$, v_a is convex and strictly above the identity; (ii) on $[\underline{x}_a, \overline{x}_a]$, v_a is equal to the identity; and (iii) on $(\overline{x}_a, 1]$, v_a is concave and strictly below than the identity. Further, v_a is increasing and $\forall x \in [0,1]$, $\max\{\sup_{x\in[0,\overline{x}_a]} \underline{\partial} v_a(x), \sup_{x\in[\underline{x}_a,1]} \overline{\partial} v_a(x)\} \leq 1$.

where $\underline{\partial} v_a$ and $\overline{\partial} v_a$ denote, respectively, the sub- and supergradient of v_a on $[0,\overline{x}_a]$ and $[0,\underline{x}_a]$.[12]

Proposition 11. v_a is of class $\mathscr{C}^2$ everywhere except possibly at $\overline{x}_a$ where it might not be differentiable. Moreover, (i) $v'_a(0) = 0$, and (ii) v_a is not differentiable at $\overline{x}_a$ only if (a) $\lim_{x\to\overline{x}_a^-} v'_a(x) \geq \lim_{x\to\overline{x}_a^+} v'_a(x)$, (b) $b(\overline{x}_a) = 0$, and (c) if $\overline{x}_a < 1$, then $b(1) > 0$.

[12] That is, $\underline{\partial} v_a(x) := \{p \mid v_a(x') - v_a(x) \geq p(x'-x), \forall x' \in [0,\overline{x}_a]\}$ and $\overline{\partial} v_a(x) := \{p \mid v_a(x') - v_a(x) \leq p(x'-x), \forall x' \in [\underline{x}_a, I]\}$.

B.1.1. Proof of Proposition 10 (Value Function is Convex-Concave)

Proof. By Proposition 2, the unique viscosity solution v is continuous. Hence, let $X^> := \{x \in [0,1] \mid v_a(x) > x\}$ and $X^< := \{x \in [0,1] \mid v_a(x) < x\}$, $X^= := [0,1] \setminus (X^> \cup X^<)$. Given that $F_a(x, v_a(x), M) := r(v_a(x) - x) - b(x)M - \frac{1}{2rc}[M_+]^2$ and $b \geq 0$, by virtue of v_a being a subsolution (resp. supersolution), for any interval $I \subseteq X^>$ (resp. $I \subseteq X^<$) we have that $M > 0$ (resp. $M < 0$) for all $x \in I$ and all $(\psi, M) \in \overline{J}^{2,+}_{[0,1]} v_a(x)$ (resp. $\overline{J}^{2,-}_{[0,1]} v_a(x)$). Further, we note that, on $X^=$, v_a is linear. As by Alvarez et al. (1997, Lemma 1), for any convex and open subset $I \subseteq X^> \cup X^=$ (resp. $I \subseteq X^< \cup X^=$), v_a is convex (resp. concave) on I.

We now show that for any element x in an interval $I \subseteq X^>$, its subgradient is such that $\max \underline{\partial} v_a(x) < 1$. As v_a is convex on I, its non-empty-, compact, convex-valued, and non-decreasing. If $\max \underline{\partial} v_a(x) \geq 1$, then this is not the case, then we have that $v_a(x') \geq v_a(x) + x' - x > x'$ for any $x' \in I$ such that $x' > x$. By continuity of v_a, $[x, 1] \subseteq X^>$ and we obtain $v_a(1) > 1$. However, as v_a is a subsolution we must have that $0 \geq \min\{F_a(1, v_a(1), M), B(1, p)\} = B(1, p)$ for any $(p, M) \in \overline{J}^{2,+}_{[0,1]} v_a(1)$. And, by convexity of v_a on $[x, 1]$ and the fact that $\max \underline{\partial} v_a(x) \geq 1$, we have that $(1, 0) \in \overline{J}^{2,+}_{[0,1]} v_a(1)$, resulting in $B(1, p) = 1 > 0$, a contradiction. An analogous argument holds to show that the supergradient of v_a at any point x of an open interval $I \subseteq X^<$ satisfies $\max \overline{\partial} v_a(x) < 1$.

The bound on the supergradient of v_a implies that, if $x \in X^<$, it must be that $\forall x' \in [x, 1]$, $v_a(x') < v_a(x) + x' - x < x'$ and thus $x' \in X^<$. We then note that $\sup X^> \leq \inf X^{\leq}$, as if $x \in X^=$ and for some $x' \geq x$ we have $x' \in X^>$, we obtain a contradiction from the fact that $v_a(x') - v_a(x) = v_a(x') - x > x' - x$. Hence there is $\underline{x}_a, \overline{x}_a \in [0,1]$ such that $[0, \underline{x}_a) = X^>$, $[\underline{x}_a, \overline{x}_a] = X^=$, and $(\overline{x}_a, 1] = X^>$, with $X^<$ and $X^>$ potentially empty.

Next, we clarify that, in fact, $X^>, X^= \neq \emptyset$ (noting $X^>$ is an open set in $[0,1]$), by showing that $0 \in X^>$. Suppose, to the contrary, that $v_a(0) = 0$ (and thus $X^{\leq} = [0,1]$, with v_a being concave on $[0,1]$). If there is some $x' \in [0,1]$ such that $v(x') > 0$, let $p := \frac{v_a(x')}{x'} > 0$. As v is concave, we have that $v_a(x) = v_a(x) - v_a(0) \geq p(x - 0) = p \cdot x > \frac{p}{2}(x + x^2)$ for all $x \in [0, x']$, and so $(\frac{p}{2}, p) \in \overline{J}^{2,-}_{[0,1]} v_a(0)$ and $\max\{F_a(0, v_a(0), p), B(0, \frac{p}{2})\} < 0$, contradicting the fact that v_a is a supersolution. If there is no such x', then $v_a \equiv 0$ and $(0, -1) \in \overline{J}^{2,-}_{[0,1]} v_a(1/2)$, with $F(1/2, v_a(1/2), -1) = -1/2 < 0$, again contradicting the fact that v_a is a supersolution.

Our last step is to show that v_a is increasing. First we note that, by convexity of v_a, $\max\overline{\partial}v_a(x) \le \min\overline{\partial}v_a(x')$ for any $x,x' \in X^{\geq}$ such that $x' > x$. Supposing, for the purpose of contradiction, that $\max\overline{\partial}v_a(0) < 0$, implies that for any $x \in (0,\overline{x}_a]$, $0 > v_a(0) - v_a(x)$. But then, letting $p := \frac{v_a(x)-v_a(0)}{x} < 0$, we have $(p,0) \in \overline{J}^{2,+}_{[0,1]}v_a(0)$, which results in $\min\{F(0,v_a(0),0),B(0,p)\} > 0$, a contradiction to the fact that v_a is a subsolution. As, symmetrically on $[\underline{x}_a,1]$, v_a is concave and thus $\min\underline{\partial}v_a(x) \ge \max\underline{\partial}v_a(x')$ for $x,x' \in X^{\leq}$, it suffices to show that $0 \in \underline{\partial}v_a(1)$. Suppose to the contrary that for some $x \in [\underline{x}_a,1)$, $1 \ge v_a(x) > v_a(1)$. We then have that $p := \frac{v_a(1)-v_a(x)}{1-x} < 0$ and $(p,0) \in \overline{J}^{2,-}_{[0,1]}v_a(1)$, which implies $\max\{F_a(1,v_a(1),0),B(1,p)\} < 0$, now a contradiction to v_a being a supersolution. $\square$

B.1.2. Proof of Proposition 11 (Value Function is $\mathscr{C}^2$, except possibly at a point)

Proof. **v'' exists a.e.:** From Proposition 10, we have that $\exists \underline{x}_a,\overline{x}_a \in [0,1]$ such that $\underline{x}_a \le \overline{x}_a$ and v_a is convex on $[0,\overline{x}_a]$ and concave on $[\underline{x}_a,1]$. By Alexandrov theorem, v_a is twice differentiable a.e. on $[0,1]$, and so it has left- and right-derivatives everywhere, denoted by $v'_{a,-}$ and $v'_{a,+}$ respectively.

No convex kinks: Take any $x' \in [0,1]$. Suppose by contradiction that $v'_{a,-}(x') < v'_{a,+}(x')$ and fix $p \in (v'_{a,-}(x'),v'_{a,+}(x'))$. Note that, for any fixed $M > 0$, $(p,M) \in \overline{J}^{2,-}_{[0,1]}v_a(x')$.[13] However, for large enough M, $F_a(x',v_a(x'),M) = r(v_a(x')-x') - b(x')M - \frac{1}{2rc}M_+^2 < 0$, contradicting the fact that v_a is a supersolution.

At most one concave kink at $\overline{x}$: Now take any $x' \in [0,1]$. Again suppose by contradiction that $v'_{a,-}(x') > v'_{a,+}(x')$ and fix $p \in (v'_{a,+}(x'),v'_{a,-}(x'))$. By a similar argument as before, we have, for any fixed $M > 0$, $(p,-M) \in \overline{J}^{2,+}_{[0,1]}v_a(x')$. For $b(x') > 0$ and large enough M, $F_a(x',v_a(x'),-M) = r(v(x')-x') + b(x')M > 0$, which contradicts v_a being a subsolution. For $b(x') = 0$ and $x' \in (0,1)$, we have $F_a(x',v(x'),-M) = r(v_a(x')-x') \le 0$. As v_a is player A's value function, whenever $b(x') = 0$, the player can attain at least $v_a(x') \ge x'$ by setting the control to zero. Hence, we must have $v_a(x') = x'$. As $v_a(x') = x' \iff x' \in [\underline{x}_a,\overline{x}_a]$, we obtain $v'_a(x') = 1$ for any $x' \in (\underline{x}_a,\overline{x}_a)$. Moreover, on $[0,\overline{x}_a)$, v_a is convex and $v'_{a,-}(x') \ge v'_{a,+}(x')$. Thus, there are no concave kinks except possibly at $\overline{x}$ and only if $b(\overline{x}_a) = 0$.

[13] To see this, let $f(x) := v_a(x') + p(x-x') + \frac{1}{2}M(x-x')^2$, and note that $v_a - f \ge 0$ in a neighborhood of x', therefore with x' being a local minimum of $v_a - f$.

Continuity of v_a' on $[0,1]\setminus\{\bar{x}_a\}$: On $[0,\bar{x}_a)$, v_a' exists and is monotone as v_a is convex (by Proposition 10). As v_a' is also differentiable almost everywhere, it has the intermediate value property (by Darboux theorem), which, together with monotonicity, implies that v_a' is continuous on $[0,\bar{x}_a)$. A symmetric argument applies to $(\bar{x}_a,1]$.

Existence and continuity of v_a'' on $[0,1]\setminus\{\bar{x}_a\}$: We now show that v_a'' exists and is continuous everywhere except possibly at $\bar{x}_a$. Fix $x\in[0,\bar{x}_a)$. As v_a'' exists a.e. then take any sequence $(x_n)_{n\geq 1}\subseteq[0,\bar{x}_a)$ such that $x_n\to x$ and $v_a''(x_n)$ exists for every n. Then, we have that $(v_a'(x_n),v_a''(x_n))\in J^{2,+}_{[0,1]}v_a(x_n)\cap J^{2,-}_{[0,1]}v_a(x_n)$, as this is true if and only if v_a is twice differentiable at x_n (Crandall et al. 1992, p.15). Hence, $(x_n,v_a(x_n),v_a'(x_n))\to(x,v_a(x),v_a'(x))$. Note that $\forall y\in[0,\bar{x}_a)$, we have $F_a(x,v_a(x),M)\leq 0$ for all $M\geq\overline{M}:=\max_{x\in[0,\bar{x}_a]}\sqrt{2cr}\sqrt{v_a(x)-x}$. Hence, $(v_a'(x),\overline{M})\in J^{2,+}_{[0,1]}v_a(x)$. Together with convexity of v_a on $[0,\bar{x}_a)$, this implies that $v_a''(x_n)\in[0,\overline{M}]$ for all n, and then, by compactness, $v_a''(x_n)$ has a convergent subsequence. Take any convergent subsequence and denote its limit as v_∞''. As F_a is continuous, $0=F_a(x_n,v_a(x_n),v_a''(x_n))\to F_a(x,v_a(x),v_\infty'')=0\implies(v_a'(x),v_\infty'')\in J^{2,+}_{[0,1]}v_a(x)\cap J^{2,-}_{[0,1]}v_a(x)$, ensuring that v_a is also twice differentiable at x, for any $x\in[0,\bar{x})$, and $v_a''(x)=v_\infty''$. This implies that v_a'' exists everywhere in $[0,\bar{x}_a)$. Moreover, as $v_\infty''\geq 0$ and $F_a(x,v_a(x),M')<F_a(x,v_a(x),M)$ for any $M'>M\geq 0$, we must then have v_∞'' being the limit of any convergent subsequence of $(v_a''(x_n))_{n\geq 1}$, and so, the limit of the original sequence: $v_a''(x_n)\to v_\infty''=v_a''(x)$, and we obtain that $v_a''\in\mathscr{C}^2$ on $[0,\bar{x}_a)$. A symmetric argument holds for $x\in(\bar{x}_a,1]$.

Zero derivative at 0: Suppose $v_a'(0)>0$. Then, $(v_a'(0)/2,2v_a''(0))\in J^{2,-}_{[0,1]}v_a(0)$[14] and $\max\{F_a(0,v_a(0),2v_a''(0)),B(0,v_a'(0)/2)\}<0$, contradicting the fact that v_a is supersolution.

Necessary conditions for nondifferentiability at $\bar{x}_a$: (a) and (b) follow from the there being only concave kinks and only if $b(\bar{x}_a)=0$. Moreover, if $\bar{x}_a<1$ and $b(1)=0$, then we must have v_a being convex on $[0,1]$ and linear on $[\underline{x}_a,1]$. It follows that $v_{a,-}'(\bar{x}_a)\geq 1$ (no convex kinks) and $v_{a,-}'\leq 1$, which implies $v_{a,-}'(\bar{x}_a)=1=v_{a,+}'(\bar{x}_a)$, and the argument from above extends to show that $v_a\in\mathscr{C}^2([0,1])$. Consequently, we obtain (c) by the contrapositive. □

B.2. Proof of Proposition 3 (Decreasing Control)

Proof. As $a(x)=\frac{1}{rc}v_a''(x)_+$, where v_a is the solution to (RP) given b, therefore it suffices to show that v'' is non-increasing in the convex region of v_a i.e. on $[0,\underline{x}_a]$, where $\underline{x}_a$ is as

[14]To see this, define $f(x)=v_a(0)+\frac{v_a'(0)}{2}x+v_a''(0)x^2$, noting that $f(x)\leq v_a(x)$ for small enough x.

defined in Proposition 10. Assume by contradiction that there exists $x, y \in [0, \underline{x}_a]$ such that $x > y$ and $v_a''(x) > v_a''(y)$. Then, using the fact that b is non-decreasing on this region,

$$\begin{aligned} 0 < \frac{1}{2rc}\left(v_a''(x)^2 - v_a''(y)^2\right) &= r[v_a(x) - x] - r[v_a(y) - y] - b(x)v_a''(x) + b(y)v_a''(y) \\ &\le r[v_a(x) - x] - r[v_a(y) - y] - \big(b(x) - b(y)\big)v_a''(x) \le r[v_a(x) - x] - r[v_a(y) - y] \end{aligned}$$

implying $1 < \frac{v_a(x) - v_a(y)}{x - y} = v_a'(z)$ for some $z \in (y, x)$ (mean value theorem), which contradicts that $0 \le v_a' \le 1$ (Theorem 2). □

B.3. Proof of Proposition 4 (Control is $\mathscr{C}^1$)

Proposition 4 follows from Theorem 2 and the following lemma.

Lemma 3. If $b \equiv 0$ and $\underline{x}_a^0 < 1$ the optimal control to the problem (RP) is $\mathscr{C}^1([0,1])$. If $\underline{x}_a^0 = 1$ it is $\mathscr{C}^1([0,1))$

Proof. Let v_a be a viscosity solution to (RP) on $\mathscr{O} = (0,1)$ when $b \equiv 0$ and a the associated optimal control. Define $\underline{x}_a$ as in Proposition 10. On $[0, \underline{x})$, $F(x, v_a, v_a'') = 0 \iff v_a''(x) = r\sqrt{2c}\sqrt{v_a(x) - x}$, and a is therefore continuously differentiable (even infinitely so) on this $[0, \underline{x})$. In particular, this proves the result when $\underline{x}_a = 1$. If $\underline{x}_a < 1$, then $v_a(x) = x$ on $(\underline{x}_a, 1]$ implying a is $\mathscr{C}^1$ on this interval, with $a'(x) = 0$, and $\lim_{x \downarrow \underline{x}_a} a'(x) = 0$. For the left derivative, noting that $v_a''(x)^2 = 2r^2c(v_a(x) - x)$ for any $x \in [0, \underline{x}_a)$ and $v_a''(x) > 0$, we differentiate both sides and obtain

$$a'(x) = \frac{1}{rc}v_a'''(x) = r\frac{v_a'(x) - 1}{v_a''(x)},$$

which is continuous as v is $\mathscr{C}^2$ on $[0, \underline{x}_a)$ given $b \equiv 0$ (Proposition 11). As $v_a'(x) - 1 < 0$ (Proposition 10), we have that $v_a'''(x) < 0$, and v_a' is strictly increasing and strictly concave on this interval. Hence, $v_a'(\underline{x}_a) - v_a'(x) \le v_a''(x)(\underline{x}_a - x)$, $\forall x < \underline{x}_a$. Thus,

$$0 \le \frac{v_a'(\underline{x}) - v_a'(x)}{v_a''(x)} \le \underline{x} - x.$$

As $(1 - v_a'(x))/v_a''(x) \to 0$ for $x \uparrow \underline{x}_a$, and $a_-'(\underline{x}_a) = 0$, we obtain that a is $\mathscr{C}^1$ on $[0,1]$. □

B.4. Proof of Proposition 6 (Control is Convex-Concave)

Proof. Let v_a be a viscosity solution to (RP) on $\mathscr{O} = (0,1)$ when $b \equiv 0$, and a the associated optimal control. Recall that $a \propto v_a''$. Denote by f_-' the left-derivative of f and $f^{(n)}$ its n-th order derivative. From Proposition 10, we have that $v_{a,-}' \le 1$. Owing to the regularity of the solution, and we can derive on $[0, \underline{x}_a)$:

$$v_a''(x) = r\sqrt{2c}\sqrt{v_a(x) - x} \ge 0, \qquad v_a^{(3)}(x) = r\sqrt{c/2}(v_a(x) - x)^{-1/2}(v_a'(x) - 1) = r^2 c \frac{v_a'(x) - 1}{v_a''(x)} \le 0,$$

$$v_a^{(4)}(x) = r^2 c - \frac{v_a^{(3)}(x)^2}{v_a''(x)}, \qquad v_a^{(5)}(x) = \frac{v_a^{(3)}(x)^3}{v_a''(x)^2} - 2\frac{v_a^{(3)}(x)}{v_a''(x)} v_a^{(4)}(x).$$

As $v_a'(x) < 1$ for $x \in [0, \underline{x}_a)$, $v_a^{(3)}$ is strictly negative on $[0, \underline{x}_a)$. If, for $x \in (0, \underline{x}_a)$, $v_a^{(4)}(x) = 0$, then $v_a^{(5)}(x) = \frac{v_a^{(3)}(x)^3}{v_a''(x)^2} < 0$. This implies that if, for $\tilde{x} \in (0, \underline{x}_a)$, $v_a^{(4)}(\tilde{x}) = 0$, then $v_a^{(4)}(x) \le 0$ for any $x \in (\tilde{x}, \underline{x}_a)$. That is, $\exists \tilde{x} \in [0, \underline{x}_a)$ such that v_a'' is convex on $[0, \tilde{x}]$ and concave on $[\tilde{x}, \underline{x}_a]$.

Suppose $v_{a,-}'(\underline{x}_a) = 1$. As, by Proposition 3, $\lim_{x \uparrow \underline{x}_a} v_a^{(3)}(x) = 0$, we have

$$\lim_{x \uparrow \underline{x}_a} v_a^{(4)}(x) = r^2 c - \lim_{x \uparrow \underline{x}_a} \frac{v_a^{(3)}(x)^2}{v_a''(x)} = r^2 c - r^2 c \lim_{x \uparrow \underline{x}_a} \frac{(v_a'(x) - 1)^2}{2r\sqrt{2c}(v_a(x) - x)^{3/2}}$$

$$= r^2 c - r^2 c \lim_{x \uparrow \underline{x}_a} \frac{2}{3} \frac{(v_a'(x) - 1) v_a''(x)}{r\sqrt{2c}(v_a(x) - x)^{1/2}(v_a'(x) - 1)} = r^2 c - r^2 c \lim_{x \uparrow \underline{x}_a} \frac{2}{3} = \frac{1}{3} r^2 c > 0,$$

where we used l'Hôpital's rule in the before-last line. Consequently, v_a'' is convex on $[0, \underline{x}_a]$.

Suppose now that $v_{a,-}'(\underline{x}_a) < 1$. Then $v_a^{(3)}(x) \le r^2 c \frac{v_{a,-}'(\underline{x}_a) - 1}{v_a''(0)} < 0$ for any $x \in [0, \underline{x}_a]$. As v_a'' is strictly positive, decreasing, $v_a''(x) \to 0$ as $x \to \underline{x}_a$, $v_a^{(4)}(x) < 0$ for all $x < \underline{x}_a$ close enough to $\underline{x}_a$. Hence, $\exists \tilde{x} \in [0, \underline{x}_a)$ such that v_a'' is convex on $[0, \tilde{x}]$ and concave on $[\tilde{x}, \underline{x}_a]$.

The fact that $v_{a,-}'(\underline{x}_a^-) = 1$ if $\underline{x}_a < 1$ follows from the a being $\mathscr{C}^1([0,1])$ when $\underline{x}_a < 1$ (Lemma 3). Finally, that $a_-'(\underline{x}_a) = -\infty$ follows from $v_a^{(3)}(x) = r^2 c \frac{v_a'(x) - 1}{v_a''(x)} \le r^2 c \frac{v_{a,-}'(\underline{x}_a) - 1}{v_a''(x)} < 0$. As the denominator goes to zero as x approaches $\underline{x}_a$, the result obtains. $\square$

B.5. Proof of Proposition 7 (Decoupling Equilibrium Instability)

Proof. Note that, from Theorem 1, $v_a(x) + v_b(x) \le \sup_{\alpha,\beta} r \int_0^\infty \exp(-rt)(X_t + (1 - X_t) - c_a \alpha(X_t)^2 - c_b \beta(X_t)^2) dt \le r \int_0^\infty \exp(-rt) dt = 1$. From Proposition 10, as v_a is (strictly) convex whenever $v_a(x) \ge (>) x \iff 0 \le x \le \overline{x}_a (< \underline{x}_a)$ and strictly concave elsewhere, and v_b is (strictly) convex whenever $v_b(x) \ge (>) 1 - x \iff 1 \ge x \ge \overline{x}_b (> \underline{x}_b)$, and strictly concave elsewhere, we have that $\underline{x}_a = \overline{x}_a =: \underline{x} > 0$ and $\overline{x}_a = \underline{x}_b =: \overline{x} < 1$. This implies that $a(x) = 0$ on $[\underline{x}, 1]$ and $b(x) = 0$ on $[0, \overline{x}]$. As, from Proposition 3 a is nonincreasing and b is nondecreasing, and, from a straightforward modification of the proof of Proposition 4, $v_a''' < 0$ on $[0, \underline{x}_a)$ and $v_b'''(x) > 0$ on $(\underline{x}_b, 1]$, we obtain that the optimal controls a and b are, respectively, strictly decreasing and strictly increasing. □

B.6. Proof of Proposition 8 (Inactive Benchmark and Equilibrium)

Proof. Noting that, from Proposition 7, we have that $a^*(x) > 0$ if and only if $x \in [0, \underline{x})$, we know from Theorem 2 that on $[\underline{x}, 1]$ v_a is concave. Hence, on $x \in [0, \underline{x})$, $0 = r_a(v_a(x) - x) - \frac{1}{r_a c_a}[v_a''(x)_+]^2$, and on $x \in [\underline{x}, 1]$ except at most at one point at which v_a is not twice differentiable, $0 = r_a(v_a(x) - x) - b(x)v_a''(x) \geq r_a(v_a(x) - x)$. As at the (at most one) nondifferentiability point of v_a we have a concave kink (Theorem 2), one concludes that v_a is a viscosity subsolution to the reflected problem in the inactive benchmark. As v_a^0 is a viscosity solution to the same problem (and thus a supersolution), from Lemma 1, we must have that $v_a^0 \geq v_a$. The same holds for player B. □

B.7. Proof of Lemma 2 (Deterrence Equilibria Singleton Stable Region)

Proof. We prove the lemma by contradiction. Let (a, b) be an equilibrium under parameters such that $\underline{x}_a^0 > \underline{x}_b^0$ and, for the purpose of contradiction, suppose $\underline{x} < \overline{x}$. Then, we must have that $\underline{x} < \underline{x}_a^0$ or $\underline{x}_b^0 < \overline{x}$. This is because, from Corollary 2, $\underline{x} \leq \underline{x}_a^0$ or $\underline{x}_b^0 \leq \overline{x}$, and, by assumption, $\underline{x}_a^0 > \underline{x}_b^0$.

Suppose that $\underline{x} < \underline{x}_a^0$ (the proof is symmetric for the case in which $\underline{x}_b^0 < \overline{x}$). From Theorem 2, we know that only concave kinks are permissible, and then it must be the case that $v_{a,-}'(\underline{x}) \geq v_{a,+}'(\underline{x}) = 1$. Moreover, from Proposition 8, the solution to player A's the inactive benchmark problem, v_a^0, is weakly greater than the player's equilibrium value function, $v_a \leq v_a^0$.

From Proposition 7, we know that, at an equilibrium, $b(x) = 0$ on $[0, \overline{x}] \supseteq [0, \min\{\overline{x}, \underline{x}_a^0\}]$. As

$$v_a^{0\prime\prime}(x) = r_a\sqrt{2c_a}\sqrt{v_a^0(x) - x} \geq r_a\sqrt{2c_a}\sqrt{v_a(x) - x} = v_a''(x)$$

on $[0,\min\{\overline{x},\underline{x}_a^0\}]$ and as $v_a'(0)=v_a^{0\prime}(0)=0$ (Proposition 11), it must be the case that $v_{a,-}'(\underline{x})\le v_{a,-}^{0\prime}(\underline{x})<v_{a,-}^{0\prime}(\underline{x}_a)\le 1=v_{a,+}'(\underline{x})$, a contradiction. ◻

B.8. Proof of Theorem 4 (Characterization of Deterrence Equilibria)

Let v_a^b be the unique viscosity solution to (RP) on $\mathcal{O}=(0,1)$ given $b\in\mathscr{C}^0([0,1])$ and $x_a^b:=\sup\{x\in[0,1]\mid v_a^b(x)>x\}$, and analogously define v_b^a and x_b^a for player B, given $a\in\mathscr{C}^0([0,1])$. It is straightforward to check that, for (a^*,b^*) such that $x_b^0\le x_a^{b^*}=x_b^{a^*}\le x_a^0$, then the equilibrium strategies must be given as described in the statement of Theorem 4. We then focus on showing that for any $\overline{x}\in[x_b^0,x_a^0]$, there is a unique strategy profile (a^*,b^*) such that $x_a^{b^*}=x_b^{a^*}=\overline{x}$. The proof of Theorem 4 for $\overline{x}\in(\underline{x}_b^0,\underline{x}_a^0)$ follows from the following two lemmata:

Lemma 4. For $\overline{x}\in(0,\underline{x}_a^0)$, let v_a denote the unique viscosity solution to (RP) on $\mathcal{O}=(0,\overline{x})$ when $b\equiv 0$. Then, (i) $v_a\in\mathscr{C}^5([0,\overline{x}))$, (ii) v_a is convex, (iii) v_a' is concave, (iv) $\exists\tilde{x}\in[0,\overline{x})$ such that v_a'' is convex on $[0,\tilde{x}]$ and concave on $[\tilde{x},\overline{x})$, and (v) $v_a'''(x)\to-\infty$ as $x\uparrow\overline{x}$.

Proof. That there is a unique viscosity solution to (RP) on $\mathcal{O}=(0,\overline{x})$ when $b\equiv 0$ follows from a straightforward modification of Theorem 1. Properties (i)-(v) follow from adjusting the proofs of Propositions 4 and 6. ◻

Lemma 5. Let $\overline{x}\in(0,1)$ and fix $b\in\mathscr{C}^0([0,1])$ such that (i) $b(x)=0$ for $x\le\overline{x}$, (ii) $b'(x)>0$ on $(\overline{x},1]$, (iii) $\lim_{x\downarrow\overline{x}}b'(x)=\infty$. Then, $v_a^b(x)\le x$ for $x\ge\overline{x}$.

Proof. Suppose not. Then, $v_a^b(\overline{x})>\overline{x}\implies v_a^{b\prime\prime}(\overline{x})>0$, and, by Proposition 4, v_a^b is $\mathscr{C}^3$ locally at $\overline{x}$ with $v_a'''(x)<0$ in a neighborhood of $\overline{x}$. Then, as $b(\overline{x})=0$, for small $\epsilon>0$, $F_a(\overline{x},v_a^b(\overline{x}),v_a^{b\prime\prime}(\overline{x}))=F_a(\overline{x}+\epsilon,v_a^b(\overline{x}+\epsilon),v_a^{b\prime\prime}(\overline{x}+\epsilon))=0\iff 0=(F_a(\overline{x},v_a^b(\overline{x}),v_a^{b\prime\prime}(\overline{x}))-F_a(\overline{x}+\epsilon,v_a^b(\overline{x}+\epsilon),v_a^{b\prime\prime}(\overline{x}+\epsilon)))/\epsilon=r[(v_a^b(\overline{x}+\epsilon)-v_a^b(\overline{x}))/\epsilon-1]-\frac{1}{2rc}(v_a^{b\prime\prime}(\overline{x}+\epsilon)^2-v_a^{b\prime\prime}(\overline{x})^2/\epsilon-v_a^{b\prime\prime}(\overline{x}+\epsilon)b(\overline{x}+\epsilon)/\epsilon$. Given that $\lim_{\epsilon\downarrow 0}|r[(v_a^b(\overline{x}+\epsilon)-v_a^b(\overline{x}))/\epsilon-1]-\frac{1}{2rc}(v_a^{b\prime\prime}(\overline{x}+\epsilon)^2-v_a^{b\prime\prime}(\overline{x})^2)/\epsilon|=|r(v_a^{b\prime}(\overline{x})-1)-\frac{1}{rc}v_a^{b\prime\prime}(\overline{x})v_a^{b\prime\prime\prime}(\overline{x})|<\infty$ due to v_a^b being locally $\mathscr{C}^3$, and given that $b(\overline{x}+\epsilon)/\epsilon=(b(\overline{x}+\epsilon)-b(\overline{x}))/\epsilon\to\infty$ as $\epsilon\downarrow 0$, by continuity, there is $\bar{\epsilon}:\forall\epsilon\in(0,\bar{\epsilon})$, $v_a^{b\prime\prime}(\overline{x}+\bar{\epsilon})b(\overline{x}+\epsilon)/\epsilon>2[r|v_a^{b\prime}(\overline{x})-1|+\frac{1}{rc}v_a^{b\prime\prime}(\overline{x})|v_a^{b\prime\prime\prime}(\overline{x})|]>r[(v_a^b(\overline{x}+\epsilon)-v_a^b(\overline{x}))/\epsilon-1]-\frac{1}{2rc}(v_a^{b\prime\prime}(\overline{x}+\epsilon)^2-v_a^{b\prime\prime}(\overline{x})^2/\epsilon$, and so we obtain a contradiction to $0=(F_a(\overline{x},v_a^b(\overline{x}),v_a^{b\prime\prime}(\overline{x}))-F_a(\overline{x}+\epsilon,v_a^b(\overline{x}+\epsilon),v_a^{b\prime\prime}(\overline{x}+\epsilon)))/\epsilon$. ◻

We now take care of showing that $\overline{x} \in \{\underline{x}_a^0, \underline{x}_b^0\}$ also pins-down an equilibrium as described.

If $a \equiv 0$ on $[x_b^0, 1]$, then $v_b^a = v_b^0$ on $[x_b^0, 1]$ and so b^0 is a best response to a. We then need that, if a is a best response to b^0, then $x_a^{b^0} = x_b^0$. We prove this in two steps.

First, let us show that $x_a^{b^0} \geq x_b^0$. Suppose not. Then $v_a^{b^0}(x) = x$ on $[x_a^{b^0}, x_b^0]$ and so, by Theorem 2, $v_a^{b^0} \in \mathscr{C}^2([0,1])$. Let $w_a(x) := \mathbf{1}_{x < x_a^{b^0}} v_a^{b^0}(x) + \mathbf{1}_{x \geq x_a^{b^0}} x$. As $v_a^{b^0} \in \mathscr{C}^2([0,1])$ is a viscosity solution to (RP) on $\mathscr{O} = (0,1)$ given $b = b^0$, and as $x_a^{b^0} < x_b^0$, it is straightforward to verify that w_a is a viscosity solution to (RP) on $\mathscr{O} = (0,1)$ given $b \equiv 0$. However, $x_a^{b^0} < x_b^0 < x_a^0$, which contradicts the uniqueness of the viscosity solution to the latter problem (Theorem 1).

Now, we show that $x_a^{b^0} \leq x_b^0$. Again, suppose this is not the case. Suppose not. Take any $\overline{x} \in (x_b^0, x_a^{b^0})$ and let (a^*, b^*) be the unique equilibrium such that $a^*(\overline{x}) = b^*(\overline{x}) = 0$ (Lemmata 4 and 5).

Claim 1: $v_a^{b^0} \geq v_a^{b^*}$ on $[0, x_a^{b^0}]$ and $v_a^{b^0} > v_a^{b^*}$ on $(\overline{x}, x_a^{b^0}]$. Let $\underline{w}_a(x) := \mathbf{1}_{x \leq \overline{x}} v_a^{b^*}(x) + \mathbf{1}_{x > \overline{x}} \overline{x}$. Noting that (i) $b^0 \geq b^*$ (Proposition 8) and $b^0 > 0 \equiv b^*$ on $(x_b^0, \overline{x})$, (ii) $v_a^{b^0}$ and $v_a^{b^*}$ are strictly convex on $[0, \overline{x})$, we get that $\underline{w}_a$ is a subsolution to (RP) on $\mathscr{O} = (0,1)$ given $b = b^0$, and, in particular, $\underline{w}_a \leq v_a^{b^0}$ (Lemma 1). Hence, $v_a^{b^0} \geq v_a^{b^*}$ on $[0, \overline{x}]$. And, on $(\overline{x}, x_a^{b^0}]$, by definition of these thresholds, we have $v_a^{b^0}(x) \geq x > v_a^{b^*}$.

Claim 2: $v_a^{b^0} \leq v_a^{b^*}$ on $[x_a^{b^0}, 1]$. Let $\overline{w}_a(x) := \mathbf{1}_{x \geq x_a^{b^0}} v_a^{b^0}(x) + \mathbf{1}_{x < x_a^{b^0}} x$. Noting that (i) $b^0 \geq b^*$ (Proposition 8), (ii) $v_a^{b^0}$ and $v_a^{b^*}$ are strictly concave on $(x_a^{b^0}, 1]$, we get that $\overline{w}_a$ is a subsolution to (RP) on $\mathscr{O} = (0,1)$ given $b = b^*$, and, in particular, $\overline{w}_a \leq v_a^{b^*}$. Hence, $v_a^{b^0} \leq v_a^{b^*}$ on $[x_a^{b^0}, 1]$.

However, claims 1 and 2 clearly entail a contradiction: $v_a^{b^*}(x_a^{b^0}) < v_a^{b^0}(x_a^{b^0}) \leq v_a^{b^*}(x_a^{b^0})$.

Online Appendix to
The Dynamics of Social Instability

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OA. Proof of Theorem 1 (Viscosity Characterization — Existence and Uniqueness)

OA.1. Proof of Proposition 1 (Viscosity Characterization)

Recall the control problem:

$$v(x) = \sup_{\alpha \in \mathscr{A}} \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t, \alpha_t) dt\right] \qquad \text{s.t. } dX_t = \sqrt{2\big(\alpha_t + b(X_t)\big)} dB_t - n(X_t) dK_t$$

where $f(x, a) = x - c\frac{a^2}{2}$.

Recall that we say that a discontinuous function is a viscosity solution to (RP) if its l.s.c envelope is a viscosity supersolution and its u.s.c envelope is a viscosity subsolution. The proof is standard and relies on applying the dynamic programming principle and Ito's formula — since there we could not find a derivation that exactly matches all of our assumptions, here is a direct derivation following usual steps. The closest result can be found in Lions (1986), with more regularity assumptions adapted for a more general setting. The following proof closely follows the approach in Pham (2009) (Section 4.3). The only specificity of our setup is the reflection, which has several important consequences. First and most obvious, it gives rise to the Neumann type boundary conditions. Second, it allows us to greatly relax the regularity assumptions on model primitives — in particular, it is sufficient that b is continuous as long as we can guarantee pathwise uniqueness for the SDE.

We appeal to the following version of the DPP (see e.g. Pham (2009) section 3.3.) consisting of two results:

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- For all $\alpha \in \mathscr{A}$, for all stopping time τ:

$$v(x) \geq \mathbb{E}\left[\int_0^\tau e^{-rt} f(X_t^x, \alpha_t)dt + e^{-r\tau} v(X_\tau^x)\right]$$

- For all $\epsilon > 0$, there exists $\alpha \in \mathscr{A}$ such that for all stopping time τ:

$$v(x) - \epsilon \leq \mathbb{E}\left[\int_0^\tau e^{-rt} f(X_t^x, \alpha_t)dt + e^{-r\tau} v(X_\tau^x)\right]$$

Where we use the notation X_t^x to denote the value at t of the process following the $dX_t = \sqrt{2\alpha_t + b(X_t)}dB_t - n(X_t)dK_t$ and starting from $X_0 = x$.

Supersolution Property.

Consider $x_0 \in [0,1]$ and $\varphi \in \mathscr{C}^2([0,1])$ s.t. x_0 is a global minimum of $v_* - \varphi$and without loss of generality $\phi(x_0) = v_*(x_0)$, where v_* denotes the l.s.c envelope of v. By definition, there exists a sequence x_n such that $x_n \to x_0$ and $v(x_n) \to v_*(x_0)$ as n goes to infinity. By continuity of ϕ, $\gamma_n := v(x_n) - \phi(x_n) \to v_*(x_0) - \phi(x_0) = 0$. Define h_n to be any strictly positive sequence such that $h_n \to 0$ and $\gamma_n / h_n \to 0$ as n goes to infinity. Fix an arbitrary $\eta > 0$ and define the stopping time $\tau_n := \inf\{t \geq 0, |X_t^{x_n} - x_n| > \eta\}$ (i.e. the first exit time of the process starting at x_n from a ball of size η). In turn define the stopping time $\theta_n := \tau_n \wedge h_n$.

We apply the first part of the DPP at x_n using an arbitrary constant strategy $\alpha_t \equiv a$ and stopping time θ_n:

$$v(x_n) \geq \mathbb{E}\left[\int_0^{\theta_n} e^{-rt} f(X_t^{x_n}, a)dt + e^{-r\theta_n} v(X_{\theta_n}^{x_n})\right]$$

Oserve that since x_0 is a global minimum of $v_* - \phi$, $v(x) \geq v_*(x) \geq \phi(x)$ for all $x \in [0,1]$, and by construction $v(x_n) = \phi(x_n) + \gamma_n$, hence:

$$\varphi(x_n) + \gamma_n \geq \mathbb{E}\left[\int_0^{\theta_n} e^{-rt} f(X_t^{x_n}, a)dt + e^{-r\theta_n} \varphi(X_{\theta_n}^{x_n})\right]$$

Applying Ito's formula to $e^{-\theta t}\varphi(X_\theta^x)$ yields for any $x \in [0,1]$ and stopping time θ gives:

$$\begin{aligned} e^{-r\theta}\varphi(X_t^x) = \varphi(x) &- \int_0^\theta re^{-rt}\varphi(X_t^x)dt + \int_0^\theta e^{-rt}\varphi'(X_t^x)\sqrt{2(a + b(X_t^{x_n}))}dB_t \\ &- \int_0^\theta e^{-rt}\varphi'(X_t^x)n(X_t^x)dK_t + \int_0^\theta e^{-rt}(a + b(X_t^x))\varphi''(X_t^x)dt \end{aligned}$$

Applying it with θ_n, x_n, plugging this back into our previous expression and rearranging yields:

$$\gamma_n \geq \mathbb{E}\Big[\int_0^{\theta_n} e^{-rt}\left(f(X_t^{x_n},a)+(a+b(X_t^{x_n}))\varphi''(X_t^{x_n})-r\varphi(X_t^{x_n})\right)dt$$
$$-\int_0^{\theta_n} e^{-rt}\varphi'(X_t^{x_n})n(X_t^{x_n})dK_t+\int_0^{\theta_n} e^{-rt}\varphi'(X_t^{x_n})\sqrt{2(a+b(X_t^{x_n}))}dB_t\Big]$$

Observe that the integrand in the last term $\int_0^{\theta_n}\varphi'(X_t^{x_n})(a+b(X_t^{x_n}))dB_t$ is bounded so the expectation is equal to zero. Rearranging again and diving by h_n yields:

$$\frac{\gamma_n}{h_n}+\mathbb{E}\left[\frac{1}{h_n}\int_0^{\theta_n} e^{-rt}\left(r\varphi(X_t^{x_n})-f(X_t^{x_n},a)-(a+b(X_t^{x_n}))\varphi''(X_t^{x_n})\right)dt\right]$$
$$+\mathbb{E}\left[\frac{1}{h_n}\int_0^{\theta_n}\varphi'(X_t^{x_n})n(X_t^{x_n})dK_t\right]\geq 0$$

Observe that for n high enough $\theta_n = h_n$ almost surely by continuity a.s. of the trajectories of X_t. We can use dominated convergence and the mean value theorem to get that when n goes to infinity:

$$r\varphi(x_0)-f(x_0,a)-(a+b(x_0))\varphi''(x_0)+\varphi'(x_0)n(x_0)\mathbb{1}_{x_0\in\{0,1\}}\geq 0$$

Where the last term comes by definition given $dK_0 = 0$ if $x_0 \in (0,1)$. Hence:

$$r\varphi(x_0)-\sup_{a\geq 0}\Big\{f(x_0,a)+(a+b(x_0))\varphi''(x_0)\Big\}+\varphi'(x_0)n(x_0)\mathbb{1}_{x_0\in\{0,1\}}\geq 0$$

This implies that:

- for all $x\in(0,1)$, $r\varphi(x_0)-\sup_{a\geq 0}\Big\{f(x_0,a)+(a+b(x_0))\varphi''(x_0)\Big\}\geq 0$
- for $x\in\{0,1\}$, either $r\varphi(x_0)-\sup_{a\geq 0}\Big\{f(x_0,a)+(a+b(x_0))\varphi''(x_0)\Big\}\geq 0$ or $\varphi'(x_0)n(x_0)\geq 0$ (it cannot be that both are negative since their sum is nonnegative)

From which we directly conclude that v is a supersolution to (RP).

Subsolution Property.

Consider $x_0\in[0,1]$ and $\varphi\in\mathscr{C}^2([0,1])$ s.t. x_0 is a global maximum of $v^*-\varphi$ with $\varphi(x_0) = v^*(x_0)$, where v^* denotes the u.s.c. envelope of v. Assume by contradiction that v is not a subsolution of (RP), i.e.:

- $r\varphi(x_0) - \sup_{a\geq 0}\Big\{f(x_0,a) + (a + b(x_0))\varphi''(x_0)\Big\} > 0$

- If $x_0 \in \{0,1\}$, $\varphi'(x_0)n(x_0) > 0$

Since $\varphi'(x_0)n(x_0)\mathbb{1}_{x\in\{0,1\}}$ is strictly positive on the boundary and zero away from it and $x_0 \mapsto r\varphi(x_0) - \sup_{a\geq 0}\Big\{f(x_0,a) + (a + b(x_0))\varphi''(x_0)\Big\}$ is continuous, there exists $\epsilon > 0$ and $\eta > 0$ such that for all $x \in B(x_0,\eta) \cup [0,1]$:

$$r\varphi(x) - \sup_{a\geq 0}\Big\{f(x,a) + (a + b(x))\varphi''(x)\Big\} + \varphi'(x)n(x)\mathbb{1}_{x\in\{0,1\}} \geq \epsilon$$

Then by definition of the u.s.c. envelope we can consider a sequence x_n taking values in $B(x_0,\eta) \cup [0,1]$ such that $x_n \to x_0$ and $v(x_n) \to v^*(x_0)$ as n goes to infinity. Just as before, we denote $\gamma_n := v(x_n) - \varphi(x_n) \to 0$ and h_m a strictly positive sequence such that $h_m \to 0$ and $\gamma_m/h_m \to 0$.

Define the stopping times $\tau_n := \inf\{t \geq 0, |X_t^{x_n} - x_n| > \eta'\}$ for some η' such that $0 < \eta' < \eta$ and $\theta_n := \tau_n \wedge h_n$. By the second part of the DPP stated above applied to with $\epsilon h_n/2$ and taking stopping time θ_n, there exists $\alpha^n \in \mathscr{A}$ such that:

$$v(x_n) - \frac{\epsilon h_n}{2} \leq \mathbb{E}\left[\int_0^{\theta_n} e^{-rt} f(X_t^{x_n}, \alpha_t^n)dt + e^{-r\theta_n} v(X_{\theta_n}^{x_n})\right]$$

Recall that by construction $v(x_n) = \varphi(x_n) + \gamma_n$ and $v^* \leq \varphi$, hence

$$\varphi(x_n) + \gamma_n - \frac{\epsilon h_n}{2} \leq \mathbb{E}\left[\int_0^{\theta_n} e^{-rt} f(X_t^{x_n}, \alpha_t^n)dt + e^{-r\theta_n} \varphi(X_{\theta_n}^{x_n})\right]$$

Applying Ito's formula to $e^{-r\theta_n}\varphi(X_{\theta_n}^{x_n})$ and rearranging gives:

$$\begin{aligned}\gamma_n - \frac{\epsilon h_n}{2} \leq\ & \mathbb{E}\left[\int_0^{\theta_n} e^{-rt}\left\{\left(-r\varphi(X_t^{x_n}) + f(X_t^{x_n}, \alpha_t^n) + (\alpha_t^n + b(X_t^{x_n}))\varphi''(X_t^{x_n})\right)dt - \varphi'(X_t^{x_n})n(X_t^{x_n})dK_t\right\}\right] \\ & - \mathbb{E}\left[\int_0^{\theta_n} e^{-rt}\varphi'(X_t^{x_n})\sqrt{2(\alpha_t^n + b(X_t^{x_n}))}dB_t\right]\end{aligned}$$

Since we assume b to be continuous, $\left|\varphi'(X_t^{x_n})\sqrt{2(\alpha_t^n + b(X_t^{x_n}))}\right|$ is bounded (because X_t is bounded by construction) and the last expectation term is zero. Simplifying accordingly and dividing by h_n:

$$\frac{\gamma_n}{h_n} - \frac{\epsilon}{2} + \mathbb{E}\left[\frac{1}{h_n}\int_0^{\theta_n} e^{-rt}\left\{\left(r\varphi(X_t^{x_n}) - f(X_t^{x_n}, \alpha_t^n) - (\alpha_t^n + b(X_t^{x_n}))\varphi''(X_t^{x_n})\right)dt + \varphi'(X_t^{x_n})n(X_t^{x_n})dK_t\right\}\right] \leq 0$$

By construction the term inside the integral is always greater than ϵ, hence we find:

$$\frac{\gamma_n}{h_n} + \epsilon\left(\frac{\mathbb{E}[\theta_n]}{h_n} - \frac{1}{2}\right) \leq 0$$

Observe that by construction $\frac{\mathbb{E}[\theta_n]}{h_n}$ converges to 1 when n goes to infinity (h_n goes to zero), so we obtain a contradiction and this concludes the proof.

OA.2. Proof of Proposition 2 (Existence and Uniqueness in the Control Problem)

The proof of Proposition 2 relies on a standard strategy: we first prove a comparison principle for our problem (every supersolution is above every subsolution); we then establish existence using Perron's method; the combination of those two results gives uniqueness and continuity.

We restate Lemma 1 for convenience:

> If $\overline{w}$ is a viscosity supersolution and $\underline{w}$ is a viscosity subsolution to (RP), then $\overline{w} \geq \underline{w}$ in $[0,1]$.

We first outline the proof structure. Take an arbitrary supersolution $\overline{w}$ (l.s.c without loss) and an arbitrary subsolution $\underline{w}$ (u.s.c without loss), and assume towards a contradiction that $\sup_{x\in[0,1]} \underline{w}(x) - \overline{w}(x) > 0$. Observe that the supremum is attained and denote by x^* a point at which it is. The steps of the proof are:

1. Show that the supremum cannot be attained inside the domain, i.e. $x^* \notin (0,1)$. This is done using standard approximation techniques for viscosity solutions, more specifically the canonical dedoubling variable technique and Ishii's lemma.

2. Consider $x^* = 0$.

(a) Show that $\overline{w}$ is non-increasing in some neighborhood to the right of 0.

(b) Show that if either $\underline{w}(0) > 0$ or $b(0) > 0$, then $\underline{w}$ is non-decreasing in some neighborhood to the right of 0

(c) Observe that $\overline{w}(0) < 0$ implies $b(0) > 0$, so if $\underline{w}(0) > \overline{w}(0)$ then either either $\underline{w}(0) > 0$ or $b(0) > 0$, therefore by the previous point $\underline{w}$ is non-decreasing in some neighborhood to the right of 0.

(d) This yields a contradiction because if $\underline{w}$ is non-decreasing and $\overline{w}$ is non-increasing in a neighborhood of 0 to the right, the supremum cannot be attained at 0 (this would only be possible if it was also attained in the interior, which we know it is not).

3. Consider $x^* = 1$

(a) Show that $\overline{w}$ is non-decreasing in some neighborhood to the left of 1.

(b) Show that if either $\underline{w}(1) > 1$ or $b(1) > 0$, then $\underline{w}$ is non-increasing in some neighborhood to the left of 0

(c) Observe that $\overline{w}(1) < 1$ implies $b(1) > 0$, so if $\underline{w}(1) > \overline{w}(1)$ then either either $\underline{w}(1) > 1$ or $b(1) > 0$, therefore by the previous point $\underline{w}$ is non-increasing in some neighborhood to the left of 1.

(d) This yields a contradiction because if $\overline{w}$ is non-increasin and $\underline{w}$ is non-decreasing in a neighborhood of 1 to the left, the supremum cannot be attained at 1 (this would only be possible if it was also attained in the interior, which we know it is not).

4. This gives a contradiction, so we conclude $\sup_{x\in[0,1]} \underline{w}(x) - \overline{w}(x) \leq 0$, which entails $\underline{w} \leq \overline{w}$ for all x.

It is interesting to note that this is very close in spirit to standard proofs in the literature (see Crandall et al. (1992)) but because of the presence of non-Lipschitz terms in the HJB equation parts of the canonical approximation methods will fail. Hence we have to appeal to arguments that are ad hoc to the structure of the problem (which would generally be quite ill-conditioned). We now detail the steps of the proof.

Proof. Consider $\overline{w}$ a supersolution to (RP) and $\underline{w}$ a subsolution to (RP). Without loss of generality, we can assume $\overline{w}$ to be l.s.c and $\underline{w}$ to be u.s.c — indeed, if we do not, the proof goes through the same way for the l.s.c (resp. u.s.c) envelope of $\overline{w}$ (resp. $\underline{w}$), in turn giving the same result since $\overline{w}(x) \geq \overline{w}_*(x) \geq \underline{w}^*(x) \geq \underline{w}(x)$.

Assume by contradiction that $\sup_{x\in[0,1]} \underline{w}(x) - \overline{w}(x) > 0$. This supremum is attained (since $\underline{w} - \overline{w}$ is u.s.c) and we denote x^* a point which attains it.

1. We first show a maximum principle result: the supremum of $\underline{w} - \overline{w}$ cannot be attained in the interior of the domain, i.e. $x^* \in \{0,1\}$.

Assume towards a contradiction that $x^* \in (0,1)$. Define:

$$M_\alpha := \sup_{x,y\in[0,1]} \underline{w}(x) - \overline{w}(y) - \frac{\alpha}{2}|x-y|^2$$

this supremum is attained and we denote (x_α, y_α) a point at which it is. Clearly $M_\alpha \geq \underline{w}(x^*) - \overline{w}(x^*) > 0$; furthermore $\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0$ and $\lim_{\alpha\to\infty} M_\alpha = \underline{w}(x^*) - \overline{w}(x^*)$ (this is a general result, see for instance Lemma 3.1. in CIL).

Let $f(x,y) := \underline{w}(x) - \overline{w}(y)$. Using Ishii's Lemma (Theorem 3.2. in CIL), we know that if $\psi \in \mathscr{C}^2([0,1]^2)$ is such that $(\hat{x},\hat{y})$ is a local maximum of $f-\psi$, then for each $\epsilon > 0$ there exist $Y, X \in \mathbb{R}$ such that:

- $(D_x\psi(\hat{x},\hat{y}), X) \in \overline{J}^{2,+}_{\mathcal{O}} \underline{w}(\hat{x})$, i.e. there exists $\underline{\varphi} \in \mathscr{C}^2$ such that $\hat{x}$ is a local minimum of $\underline{w} - \underline{\varphi}$ and $\underline{\varphi}'(\hat{x}) = D_x\psi(\hat{x},\hat{y})$, $\underline{\varphi}''(\hat{x}) = X$
- $(-D_y\psi(\hat{x},\hat{y}), Y) \in \overline{J}^{2,-}_{\mathcal{O}} \overline{w}(\hat{y})$, i.e. there exists $\overline{\varphi} \in \mathscr{C}^2$ such that $\hat{y}$ is a local maximum of $\overline{w} - \overline{\varphi}$ and $\overline{\varphi}'(\hat{y}) = -D_y\psi(\hat{x},\hat{y})$, $\overline{\varphi}''(\hat{y}) = Y$
- And we have:

$$-\left(1 + ||D^2\psi(\hat{x},\hat{y}||)\right) I_2 \leq \begin{pmatrix} \underline{\varphi}''(\hat{x}) & 0 \\ 0 & -\overline{\varphi}''(\hat{y}) \end{pmatrix} \leq D^2\psi(\hat{x},\hat{y}) + \epsilon\left(D^2\psi(\hat{x},\hat{y})\right)^2$$

Hence for any $\alpha > 0$, we can take $\epsilon = 1/\alpha$ and apply this result at (x_α, y_α) with $\psi_\alpha(x,y) := \frac{\alpha}{2}|x-y|^2$, i.e. there exists $\underline{\varphi}_\alpha, \overline{\varphi}_\alpha$ appropriate test functions for $\underline{w}, \overline{w}$ respectively at x_α, y_α such that:

$$\overline{\varphi}'_\alpha(x_\alpha) = D_x\psi(x_\alpha, y_\alpha) = \alpha(x_\alpha - y_\alpha)$$

$$\underline{\varphi}'_\alpha(x_\alpha) = -D_y\psi(x_\alpha, y_\alpha) = \alpha(x_\alpha - y_\alpha)$$

$$-3\alpha I_2 \leq \begin{pmatrix} \underline{\varphi}''_\alpha(x_\alpha) & 0 \\ 0 & -\overline{\varphi}''(y_\alpha) \end{pmatrix} \leq 3\alpha \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Which, in particular, implies that $\underline{\varphi}''_\alpha(x_\alpha) \leq \overline{\varphi}''_\alpha(y_\alpha)$ for all $\alpha > 0$.

Since $x^* \in (0,1)$, the supersolution and subsolution properties entail that for any α

$$F(x_\alpha, \underline{w}(x_\alpha), \underline{\varphi}''_\alpha(x_\alpha)) \leq 0 \leq F(y_\alpha, \overline{w}(y_\alpha), \overline{\varphi}''_\alpha(y_\alpha))$$

Or explicitly:

$$r\underline{w}(x_\alpha) - rx_\alpha - b(x)\underline{\varphi}''_\alpha(x_\alpha) - \frac{1}{2rc}[\underline{\varphi}''_\alpha(x_\alpha)_+]^2 \le 0 \le r\overline{w}(y_\alpha) - ry_\alpha - b(y)\overline{\varphi}''_\alpha(y_\alpha) - \frac{1}{2rc}[\overline{\varphi}''_\alpha(y_\alpha)_+]^2$$

Rearranging yields:

$$r(\underline{w}(x_\alpha) - \overline{w}(y_\alpha)) - r(x_\alpha - y_\alpha) \le b(x)\left(\underline{\varphi}''_\alpha(x_\alpha) - \overline{\varphi}''_\alpha(y_\alpha)\right) + \frac{1}{2rc}\left([\underline{\varphi}''_\alpha(x_\alpha)]^2 - [\overline{\varphi}''_\alpha(y_\alpha)_+]^2\right) \le 0$$

Which then implies:

$$r(\underline{w}(x_\alpha) - \overline{w}(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2) - r(x_\alpha - y_\alpha) \le 0$$

Taking the limit in the left hand side yields:

$$\underline{w}(x^*) - \overline{w}(x^*) \le 0$$

Which contradicts our premise.

Therefore if $\sup_{x\in[0,1]} \underline{w}(x) - \overline{w}(x) > 0$, the supremum can only be attained on the boundary i.e. $x^* \in \{0,1\}$.

2. Consider the case $x^* = 0$.

(a) We first prove that $\overline{w}$ is non-increasing in some right neighborhood of 0. By definition of the second-order subjet, it is sufficient to show that for all $(p,M) \in \overline{J}^{2,-}_{[0,1]}\overline{w}(0)$, $p \le 0$. Assume by contradiction that there exists $(p,M) \in \overline{J}^{2,-}_{[0,1]}\overline{w}(0)$ such that $p > 0$. Consider any p' such that $0 < p' < p$ and an arbitrary $M' > 0$. There must exist some neighborhood of 0 (to the right) such that $px + \frac{1}{2}Mx^2 \le p'x + \frac{1}{2}M'x^2$ (the first order terms dominate for x small enough). Therefore as $x \to 0$:

$$\overline{w}(x) \ge \overline{w}(0) + px + \frac{1}{2}Mx^2 + o(x^2) \ge \overline{w}(0) + p'x + \frac{1}{2}M'x^2 + o(x^2)$$

Hence $(p',M') \in J^{2,-}_{[0,1]}\overline{w}(0)$. Since this holds (close enough to zero) for M' arbitrarily large, we get a contradiction since $B(0,p') < 0$ and $F(0,\overline{w}(0),M') < 0$ for M' large enough.

(b) We claim that if either $\underline{w}(0) > 0$ or $b(0) > 0$, then $\underline{w}$ has to be non-decreasing in some neighborhood of 0. Again it is sufficient to show that for all $(p,M) \in J^{2,+}_{[0,1]}\underline{w}(0)$, $p \ge 0$. Assume by contradiction that there exists $(p,M) \in \overline{J}^{2,+}_{[0,1]}\underline{w}(0)$ with $p < 0$. Take any p' such that $p < p' < 0$. For an arbitrary $M' < 0$, there must exist some neighborhood of 0 (to the right) such that $px + \frac{1}{2}Mx^2 \le p'x + \frac{1}{2}M'x^2$ (the second order terms vanish faster as x goes to zero), therefore:

$$\underline{w}(x) \le \underline{w}(0) + px + \frac{1}{2}Mx^2 + o(x^2) \le \underline{w}(0) + p'x + \frac{1}{2}M'x^2 + o(x^2) \quad \text{as } x \to 0$$

This means that $(p', M') \in J^{2,+}_{[0,1]}\underline{w}(0)$. Observe that $F(0, \underline{w}(0), M') = r\underline{w}(0) - b(0)M' > 0$ when either $\underline{w}(0) > 0$ or $b(0) > 0$. Hence this is a contradiction since $B(0,p) > 0$ and $F(0, \underline{w}(0), 0) > 0$.

(c) If $\overline{w}(0) < 0$, it must be that $b(0) > 0$. Indeed, if $b(0) = 0$, then by continuity for all $\varepsilon > 0$ we can find $x_\varepsilon > 0$ such that $0 \le b(x_\varepsilon) < \varepsilon$. For any ϵ, select arbitrarily $(p_\varepsilon, M_\varepsilon) \in \overline{J}^{2,-}_{[0,1]}\underline{w}(x_\varepsilon)$. We have for all $\varepsilon > 0$:

$$\begin{aligned} 0 \le F(x_\varepsilon, \overline{w}(x_\varepsilon), M_\varepsilon) &= r\overline{w}(x_\varepsilon) - rx_\varepsilon - b(x_\varepsilon)M_\varepsilon - \frac{1}{2rc}M_{\varepsilon+}^2 \\ &\le r\overline{w}(0) - b(x_\varepsilon)M_\varepsilon \end{aligned}$$

Since $b(x_\varepsilon) < \varepsilon$, this must imply that $M_\varepsilon < -\frac{r\overline{w}(0)}{\varepsilon}$, in other words as we get close enough to zero the second order terms in the subjets are bounded above by an arbitrarily negative constant. This is clearly a contradiction since it essentially would mean that $\overline{w}$ is locally bounded above by an arbitrarily concave paraboloid as we get closer to zero. To make this point formal, define $M'_\varepsilon := M_\varepsilon + \varepsilon$; from the previous point $(p_\varepsilon, M'_\varepsilon) \notin \overline{J}^{2,-}_{[0,1]}\underline{w}(x_\varepsilon)$, i.e. by definition:

$$\overline{w}(x) < \overline{w}(x_\varepsilon) + p_\varepsilon(x - x_\varepsilon) + \frac{1}{2}M'_\varepsilon(x - x_\varepsilon)^2 + o((x - x_\varepsilon)^2) \quad \text{as } x \to x_\varepsilon$$

In other words, defining $\varphi_\varepsilon(x) := \overline{w}(x_\varepsilon) + p_\varepsilon(x - x_\varepsilon) + \frac{1}{2}M'_\varepsilon(x - x_\varepsilon)^2$, x_ε is *not a local minimum* of $\overline{w} - \varphi_\varepsilon$. But by construction since $M'_\varepsilon \to -\infty$ and $x_\varepsilon \to 0$ as ε goes to zero, $\varphi_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} \mathbb{1}_{x \neq 0} \times (-\infty)$ i.e. the function that has value 0 at 0 and minus infinity everywhere else, hence $\liminf_{x \to 0} \overline{w}(x) = -\infty < \overline{w}(0)$ contradicting that $\overline{w}$ is l.s.c.

(d) This entails that if $\underline{w}(0) > \overline{w}(0)$, then $\underline{w}$ is non-decreasing in some neighborhood of 0 to the right — because either $\overline{w}(0) \ge 0$ which implies $\underline{w}(0) > 0$ or $\overline{w}(0) < 0$ which implies $b(0) > 0$. Therefore, we have that in some neighborhood of 0 $\overline{w}$ is non-increasing and $\underline{w}$ is non-decreasing, which directly contradicts the fact that the supremum of $\underline{w} - \overline{w}$ is reached at 0 and not in the interior.

3. The only remaining possibility is $x^* = 1$. The derivations are symmetrical to the previous case.

(a) First prove that $\overline{w}$ is non-decreasing in a neighborhood of 0 to the left. Assume by contradiction that there exists $(p,M) \in \overline{J}^{2,-}_{[0,1]}\overline{w}(1)$ such that $p < 0$. Consider any p' such that $p < p' < 0$ and an arbitrary $M' > 0$. As $x \to 0$:

$$\begin{aligned}\overline{w}(x) &\geq \overline{w}(1) + p(x-1) + \frac{1}{2}M(x-1)^2 + o((x-1)^2) \\ &\geq \overline{w}(1) + p'(x-1) + \frac{1}{2}M'(x-1)^2 + o((x-1)^2)\end{aligned}$$

since $x - 1 \leq 0$. Hence $(p', M') \in J^{2,-}_{[0,1]}\overline{w}(0)$. Since this holds (close enough to zero) for M' arbitrarily large, we get a contradiction since $B(0,p') < 0$ and $F(0,\overline{w}(0),M') < 0$ for M' large enough.

(b) Assume that either $\underline{w}(1) > 1$ or $b(1) > 0$. Assume by contradiction that there exists $(p,M) \in \overline{J}^{2,+}_{[0,1]}\underline{w}(1)$ with $p > 0$. Take any p' such that $0 < p' < p$. For an arbitrary $M' < 0$, there must exist some neighborhood of 1 (to the left) such that $p(1-x)+\frac{1}{2}M(1-x)^2 \leq p'x + \frac{1}{2}M'x^2$ (the second order terms vanish faster as x goes to zero), therefore:

$$\begin{aligned}\underline{w}(x) &\leq \underline{w}(1) + p(1-x) + \frac{1}{2}M(1-x)^2 + o((1-x)^2) \quad \text{as } x \to 0 \\ &\leq \underline{w}(1) + p'(1-x) + \frac{1}{2}M'(1-x)^2 + o((1-x)^2) \quad \text{as } x \to 0\end{aligned}$$

i.e. $(p',M') \in J^{2,+}_{[0,1]}\underline{w}(1)$. Observe that for $M' < 0$, $F(1,\underline{w}(1),M') = r\underline{w}(1) - b(1)M' > 0$ when either $\underline{w}(1) > 0$ or $b(1) > 0$. Hence this is a contradiction since $B(1,p) > 0$ and $F(1,\underline{w}(1),1) > 0$. Therefore if either $\underline{w}(1) > 1$ or $b(1) > 0$, $\underline{w}$ is non-increasing in a neighborhood of 1.

(c) Show that if $\overline{w}(1) < 1$, we must have $b(1) > 0$. assume towards a contradiction that $b(1) = 0$, the argument is exactly symmetrical to the one we made at zero since this implies $\overline{w}(1) - 1 < 0$, therefore continuity of b implies that the any second-order subjet of $\overline{w}$ at x must be bounded above by some quantity that goes to $-\infty$ when x goes to 1; this means that $\overline{w}$ is locally bounded above by some paraboloid that we can make arbitrarily concave as x goes to 1, hence $\liminf_{x\to 1}\overline{w}(x) = -\infty < \overline{w}(1)$ contradicting that $\overline{w}$ is l.s.c.

(d) This entails that if $\underline{w}(1) > \overline{w}(1)$, then $\underline{w}$ is non-increasing in some neighborhood of 1 to the left — because either $\overline{w}(1) \geq 1$ which implies $\underline{w}(1) > 1$ or $\overline{w}(1) < 1$ which implies $b(1) > 0$. Therefore, we have that in some neighborhood of 1 $\overline{w}$ is non-decreasing and $\underline{w}$ is non-increasing, which directly contradicts the fact that the supremum of $\underline{w} - \overline{w}$ is reached at 1 and not in the interior.

Putting those points together yields a contradiction. Therefore, we conclude that

$$\sup_{x \in [0,1]} \underline{w}(x) - \overline{w}(x) \leq 0$$

which entails $\overline{w}(x) \geq \underline{w}(x)$ for all $x \in [0,1]$ and concludes the proof.

$\square$

Lemma 6 (Existence — Perron's Method)**.** If the comparison principle holds for (RP), and if there is a subsolution $\underline{w}$ and a supersolution $\overline{w}$ that satisfy the boundary conditions (in the viscosity sense), then:

$$\hat{w}(x) := \sup\left\{ w(x) : \underline{w} \leq w \leq \overline{w} \text{ and } w \text{ is a subsolution of (RP) } \right\}$$

is a solution of (RP).

This is standard and can be directly applied from e.g. Crandall et al. (1992). Furthermore, things are particularly easy in our setup since we can exhibit and explicit supersolution (take $\overline{w}(x) := 1$ for all x) and an explicit subsolution (take $\underline{w}(x) := 0$ for all x), directly giving existence of a solution.