A Mean Field Game Model for the Dynamics of Cities

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Modelling the Dynamics of Cities

- Agglomeration forces
  commuting, amenities, rents, externalities,...
- Limited Control
  $\rightarrow$ Is location fixed or chosen ? $\rightarrow$ **Both.**
- Endogenous dynamics ?
  $\rightarrow$ No Steady State assumptions
- Can we generate complex dynamics with a very sparse model ?
  $\rightarrow$ Commuting Costs (Spatial Labor Market)
  $\rightarrow$ Rents / Congestion

Dynamically Endogenous Characteristics

- A broader modelling question : **multiple time scales**
  - $\rightarrow$ Locally in time, some characteristic is *taken as* **exogenous**
    $\Rightarrow$ **"Instantaneous" Equilibrium**
  - $\rightarrow$ ...But becomes **endogenous** over time (*at a cost*)

- Equilibrium payoffs :
  - $\rightarrow$ Contingent on the *whole distribution* of types and its *dynamics*

- A general approach :
  - $\rightarrow$ Instantenous Matching Equilibrium $\Rightarrow$ Optimal Transport
  - $\rightarrow$ Continuous Time Dynamics $\Rightarrow$ Mean Field Games

Alternative Interpretations

- City Dynamics
  - → Populations : Workers and Firms
  - → Types : Geographical Location
  - → Instaneous Matching : Labor Market
  - → Congestion : Rents

- Labor Market
  - → Populations : Workers and Firms
  - → Types : Skills and Productive type
  - → Instaneous Matching : Labor Market
  - → Congestion : Demand Effect (sector size)

- (Intermediate) Goods Market
  - → Populations : Sellers and Buyers
  - → Types : Goods Type
  - → Instaneous Matching : Goods Market
  - → Congestion : ?

Related Litterature

- Matching and Optimal Transport :
  - → OT : Monge (1781), Kantorovitch (1942), Brenier (1991)
  - → Well known connection : Chiappori, McCann, Nesheim (2010) ; Chiappori, Salanié (2016) ; Villani (2003,2008) ; Carlier, Ekeland (2016) ; Gallichon (2016) ; Santambrogio (2015)
- Mean Field Games :
  - → Lasry, Lions (2006) ; growing litterature
  - → in Applied Maths : Cardaliaguet, Lasry, Lions, Porretta (2012) ; Cardaliaguet, Graber, Porretta, Tonon (2015) ; Benamou, Carlier, Di Marino, Nenna (2019)
  - → in Economics : Heterogeneous Agents Model ; Achdou, Buera, Lasry, Lions, Moll (2014)
- Sinkhorn/IPFP algorithm :
  - → Sinkhorn (1968), Cuturi (2013), Peyre (2015), Benamou, Carlier, Cuturi, Nenna, Peyre (2015), Benamou, Carlier, Di Marino, Nenna (2019)
  - → In Economics : Berry-Levinsohn-Pakes (1995) ; Gravity Models ; Choo, Siow (2006) ; Chiappori, Salanié (2016) ; Galichon, Salanié (2020).
- Urban and Geographical Economics : usually a different perspective...

Model Structure I : Setup

- Time : $t \in [0, T]$
- Space : Ω (typically $\mathbb{T}^d$ or $\mathbb{R}^d$, $d \leq 2$)
- Two populations (continuum, mass 1) :
  - $\rightarrow$ Residents/Workers with density $m_1(t, \cdot)$
  - $\rightarrow$ Firms with density $m_2(t, \cdot)$
  - $\rightarrow$ Initial densities m_1^0, m_2^0 are given.
- Unknown dynamics

Model Structure II : Instantaneous Interactions

- Labour Market
  - $\rightarrow$ In equilibrium at any time t for given densities $m_1(t,\cdot), m_2(t,\cdot)$
  - $\Rightarrow$ Wages & Profits
- Land Market
  - $\rightarrow$ In equilibrium at any time t for given densities $m_1(t,\cdot), m_2(t,\cdot)$
  - $\rightarrow$ Land owned by absentee landlords
  - $\rightarrow$ Residents and firms compete for land
  - $\Rightarrow$ Rents

Model Structure III : Dynamics

- Residents and firms solve an **Optimal Control Problem** internalizing equilibrium payoffs, cost of moving, dynamics

- Dynamics of $m_1, m_2 \Rightarrow$ Best-Response Behavior
  $\rightarrow$ Hamilton-Jacobi-Bellman Equation

- Optimal feedback $\Rightarrow$ Dynamics of m_1, m_2
  $\rightarrow$ Fokker-Planck Equation

$\Rightarrow$ Look for a **Fixed Point**

Rents

- Rents : $R(t, x)$
- Exogenous increasing supply function : $S(R)$
- Market clearing :

$$m_1(t, x) + m_2(t, x) = S(R(t, x)) \quad \forall t, x$$

- Denote $f := S^{-1}$:

$$R(t, x) = f(m_1(t, x) + m_2(t, x))$$

Labour Market Structure

- Commuting cost : $c \in C(\Omega \times \Omega)$, $c(x,y) \geq 0 \; \forall x, y$
- Firms
  - $\rightarrow$ offer wage $w(t,y)$
  - $\rightarrow$ demand normalized to one unit of labour
- Workers solve :

$$r(t,x) := \max_{y \in \Omega} w(t,y) - c(x,y)$$

- By construction :

$$w(t,y) - r(t,x) \leq c(x,y) \quad \forall (x,y)$$

Labour Market Equilibrium I

- Denote $\gamma(x, y)$ the mass of workers from x that work at y
- Market Clearing :

$$\pi_1 \# \gamma(x) = m_1(x), \quad \pi_2 \# \gamma(y) = m_2(y)$$

- Stability (from optimizing behavior of workers and firms) :

$$\begin{aligned} w(y) - r(x) &= c(x, y) \quad \forall (x, y) \in \mathrm{spt}(\gamma) \\ w(y) - r(x) &\leq c(x, y) \quad \forall (x, y) \end{aligned}$$

→ This is the complementary slackness condition of an Optimal Transport Problem !

Labour Market Equilibrium II

- Well-known equivalence between stability and surplus maximization (cost minimization) :

$$C(m_1, m_2) = \min_{\gamma \in \Pi(m_1, m_2)} \int c(x, y) d\gamma(x, y) \qquad \text{(MP)}$$

- Dual form (Linear Programming) :

$$C(m_1, m_2) = \max_{\alpha_1, \alpha_2} \int \alpha_1(x) dm_1(x) + \int \alpha_2(y) dm_2(y) \quad \text{(KD)}$$

$$s.t. \quad \alpha_1(x) + \alpha_2(y) \leq c(x, y) \ \forall x, y$$

- Symmetrical notations :

$$\alpha_1 = -r, \quad \alpha_2 = w$$

Regularization / noise

- In practice we often consider the *regularized optimal transport problem* :

$$C^{\sigma}(m_1, m_2) = \min_{\gamma \in \Pi(m_1, m_2)} \int c(x, y) d\gamma(x, y) + \sigma \int \gamma(x, y)(\log \gamma(x, y) - 1)$$

- Note that γ has an almost closed form solution $\gamma(x, y) = \alpha_1(x)\alpha_2(y)e^{-\frac{c(x,y)}{\sigma}}$

- When $\sigma \to 0$, $C^{\sigma} \to C$: useful for numerical simulations !
Cuturi (2013), Benamou et. al. (2015, 2019)

- Can be viewed as adding **noise** in the coupling
$\rightarrow$ Link with random utility (logit) see e.g. Galichon and Salanié (2020), Chiong, Galichon, Shum (2016), Chiappori and Salanié (2016), Choo and Siow (2006).

- Position is Endogenous over time $\rightarrow$ minimize costs, taking dynamics as given.

- Movement of a resident follows a **controlled diffusion process**

$$dX_s = v_1(s, X_s)ds + \sqrt{2\nu_1}dB_s, \quad X_t = x \tag{1}$$

- Worker's Optimal Control Problem :

$$u_1(t,x) := \min_{v_1} \int_t^T \left(L_1(v_1) + R(s, X_s) + \alpha_1(s, X_s)\right) ds$$
$$\text{subject to (1)}$$

where $R(s,x) = f(m_1(s,x) + m_2(s,x))$

From Static to Dynamic : Firms

- Position is Endogenous over time $\rightarrow$ minimize costs, taking dynamics as given.

- Movement of a firm follows a **controlled diffusion process**

$$dY_s = v_2(s, Y_s)ds + \sqrt{2\nu_2}dB_s, \quad Y_t = y \tag{2}$$

- Firm's Optimal Control Problem :

$$u_2(t,x) := \min_{v} \int_t^T \left(L_2(v_2) + R(s, Y_s) + \alpha_2(s, Y_s) \right) ds$$

$$\text{subject to (2)}$$

where $R(s,y) = f(m_1(s,y) + m_2(s,y))$

Best Response : HJB

- Optimal control : characterized by solution of a PDE
  $\rightarrow$ **Hamilton-Jacobi-Bellman (HJB) equation**

- To simplify, take

$$L_i = \theta_i \frac{|v_i|^2}{2}, \quad i = 1, 2$$

- The value function $u_i(x,t)$ solves :

$$\partial_t u_i + \nu_i \Delta u_i - \frac{|\nabla u_i|^2}{2\theta_i} = -R - \alpha_i, \quad u(T, \cdot) = 0 \qquad \text{(HJB)}$$

- The optimal control is recovered as $v_i(t,x) = \frac{\nabla u_i(t,x)}{\theta_i}$

- Consider a diffusion process (individual dynamics) X_t :

$$dX_t = v(t, X_t)dt + \sqrt{2\nu}dB_t, \quad X_0 \sim m^0$$

for an arbitrary field v and initial distribution m^0

- Denote $m(x,t)$ the density (probability distribution) of X_t
  - → How to obtain the evolution of $m(t,x)$ from individual dynamics ?
- General answer : **Fokker-Planck equation**

$$\partial_t m - \nu\Delta m - div(mv) = 0, \quad m(0,\cdot) = m^0 \qquad \text{(FP)}$$

Equilibrium : the MFG system

- Optimal Control for given density dynamics $\rightarrow$ **HJB**
- Density Dynamics for given control $\rightarrow$ **Fokker-Planck**
- How to find a fixed point (i.e Nash Equilibrium) ?
  - $\rightarrow$ HJB + Fokker-Planck : system of PDEs in (u, m)
  - $\rightarrow$ This is the MFG System !
- "Canonical" **one population** MFG System (Lasry/Lions 2006)

$$\begin{cases} \partial_t u(t,x) + \nu \Delta u(t,x) - \frac{|\nabla u(t,x)|^2}{2} = -\phi(t,m) & \text{(HJB)} \\ \partial_t m - \nu \Delta m - div(m \frac{\nabla u}{\theta}) = 0 & \text{(FP)} \\ m(0,\cdot) = m^0, u(T,\cdot) = 0 \end{cases}$$

The Full MFG System

Putting it all together, we look for a solution $(u_1, u_2, m_1, m_2, \alpha_1, \alpha_2)$ where all functions are defined over (t, x)

$$\begin{cases} \partial_t u_1 + \nu_1 \Delta u_1 - \frac{|\nabla u_1|^2}{2\theta_1} = -f\left(m_1 + m_2\right) - \alpha_1 & \text{(HJB 1)} \\ \partial_t u_2 + \nu_2 \Delta u_2 - \frac{|\nabla u_2|^2}{2\theta_2} = -f\left(m_1 + m_2\right) - \alpha_2 & \text{(HJB 2)} \\ \partial_t m_1 - \nu \Delta m_1 - \text{div}(m_1 \frac{\nabla u_1}{\theta_1}) = 0 & \text{(FP 1)} \\ \partial_t m_2 - \nu \Delta m_2 - \text{div}(m_2 \frac{\nabla u_2}{\theta_2}) = 0 & \text{(FP 2)} \\ m_i(0, \cdot) = m_i^0, \ i = 1, 2 & \text{(IC)} \\ u_i(T, \cdot) = 0, \ i = 1, 2 & \text{(TC)} \end{cases}$$

Where $(\alpha_1(t, \cdot), \alpha_2(t, \cdot))$ solve the dual matching problem $\forall t$, i.e :

$$\alpha_1(t, x) + \alpha_2(t, y) \leq c(x, y) \quad \forall (t, x, y)$$

$$C(m_1(t, \cdot), m_2(t, \cdot)) = \int \alpha_1(x) m_1(t, dx) + \int \alpha_2(y) m_2(t, dy) \quad \forall t$$

Variational Approach

- Solving the MFG System directly ?
- The system has a **Variational Structure** :
  $\rightarrow$ can be rewritten formally as an infinite dimensional **constrained optimization problem**
- The equations of the MFG system are the first order (primal-dual) conditions of this problem
- Then : use the tools of infinite-dimensional convex analysis (duality theory) to prove existence and other results

The (Eulerian) Variational Problem I

- Define $F(m) := \int_0^m f(x)dx$ and :

$$\mathcal{F}(m_1, m_2) = \begin{cases} \int_\Omega F(m_1(x) + m_2(x))dx & \text{if } m_1, m_2 \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

- Go back to a general cost of motion L_i

Proposition (Variational Problem)

The MFG system is formally equivalent to :

$$\inf_{w_1, w_2, m_1, m_2} \int_0^T \int_\Omega L_1\left(x, \frac{w_1}{m_1}\right) m_1 + L_2\left(x, \frac{w_2}{m_2}\right) m_2 + \int_0^T C(m_1(t,\cdot), m_2(t,\cdot)) + \int_0^T \mathcal{F}(m_1(t,\cdot), m_2(t,\cdot))$$

subject to :

$$\begin{cases} \partial_t m_i - \nu_i \Delta m_i - div(w_i) = 0, \quad i = 1,2 \\ m_i(0,\cdot) = m_i^0, \quad i = 1,2 \end{cases}$$

A General Existence Theorem

Theorem

The relaxed Variational Problem admits at least one solution. In particular, the MFG system admits a weak solution in an appropriately defined sense.

$\rightarrow$ Relies on duality theory, calculus of variation techniques, and the theory of weak subsolution of HJB equations.
$\rightarrow$ Find solutions in a relaxed sense
$\Rightarrow$ not necessarily continuous and differentiable

The Case of Quadratic Costs

Proposition

Assume quadratic cost of motion :

$$L_i = \theta_i \frac{|v_i|^2}{2}$$

The Variational Formulation and the MFG system admit classical/strong solutions, i.e continuously differentiable solutions.

Uniqueness ?

- **Uniqueness is not guaranteed is general !**
- Potentials are only defined up to an additive constant (if I augment all wages by a constant, the equilibrium stays the same : only relative wages matter)
- In general : no hope for uniqueness.

The Space of Paths Measures

- Assume Quadratic L_i from now on.

- MFG and OT have a deep relation with Schrodinger's problem
$\rightarrow$ Dawson, Gartner (1987), Follmer (1988), Leonard (2013), Benamou et. al. (2019)

- Problem can be rewritten as an optimization in the space of probability measures on **paths** :

$$P = \mathcal{P}(C([0, T], \Omega))$$

- Define the relative entropy of measure p w.r.t q (a.k.a. Kullback-Leibler Divergence) :

$$H(p|q) = \int dp \left(\log \frac{dp}{dq} - 1 \right)$$

Entropy Minimization

- Fundamental result :

$$\inf_{\substack{v \\ \partial_t m - \frac{1}{2}\Delta m + \mathrm{div}(mv) = 0}} \int_0^T \int_\Omega \frac{|v|^2}{2} m = \inf_{\substack{Q \in P \\ e_t \# Q = m_t}} H(Q|R) - H(m_0|R_0)$$

- Minimizing energy for a given flow m_t is equivalent to minimizing the relative entropy with respect to the Wiener measure R.
- Remark : the regularized OT problem can also be rewritten as some entropy minimization

$$C^\sigma(m_1, m_2) = \sigma \min_{\gamma \in \Pi(m_1, m_2)} H(\gamma | e^{-\frac{c}{\sigma}})$$

A Second Variational Formulation

- Recall the First Variational Formulation :

$$\inf_{(m,w)} \sum_{i=1}^{2} \frac{\theta_i}{2} \int_0^T \int_\Omega \frac{|w_i|^2}{m_i} + \int_0^T (C + \mathcal{F})(m_1(t,.), m_2(t,.))\mathrm{d}t$$

- This is equivalent to :

$$\inf_{\substack{Q_1, Q_2 \in P \\ e_{0\#} Q_i = m_i^0}} \theta_1 H(Q_1|R_1) + \theta_2 H(Q_2|R_2) + \int_0^T (C + \mathcal{F})(e_{t\#} Q_1, e_{t\#} Q_2)\mathrm{d}t$$

The Sinkhorn Iterative Scaling Algorithm

- Celebrated algorithm in the Optimal Transport litterature : fast, easy to implement, scalable.
- Recently : Generalized versions $\rightarrow$ amenable to essentially any penalized entropy-minimization problem $\inf_p H(p|q) + f(p)$
- First applied to classical MFG in Benamou et. al. (2019)
- Closely related to many other litteratures : Berry-Levinsohn-Pakes (1995) ; Gravity Models ; Choo, Siow (2006) ; Chiappori, Salanié (2016) ; Galichon, Salanié (2020).

An Embedded Generalized Sinkhorn

- The beauty of the algorithm : solve everything at once (instantaneous equilibria + dynamics for both population)
- Heavy to write out... But easy to implement !
  - → Discretize the problem on a grid (time-space)
  - → Write the dual problem
  - → Alternate minimization (coordinate descent) on the dual
- Fast !

Setup

- S : discretized one-dimensional torus (circle)
- T horizon, in time steps is $N+1$ (indexed by $k = 0, ..., N$)
- θ_1, θ_2 : mobility parameters (higher $\theta \Rightarrow$ costlier to move)
- σ : "noise" in instantaneous equilibrium
- ν_1, ν_2 : diffusivity parameters for residents and firms
- the congestion/rent function is given by :

$$F(x) = \frac{ax^p}{p},$$

  higher a and p mean stronger congestion,
- Ground cost : either

  $\rightarrow$ geodesic distance (labeled as *linear*)
  $\rightarrow$ its square root (labeled *sqrt*)
  $\rightarrow$ its square distance (labeled *quadratic*).

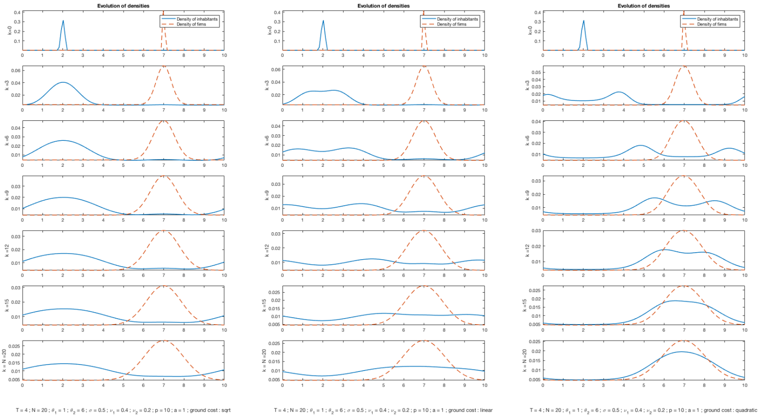


Figure 1: Effect of the ground cost

Population Asymmetry and Sensitivity to Initial Conditions

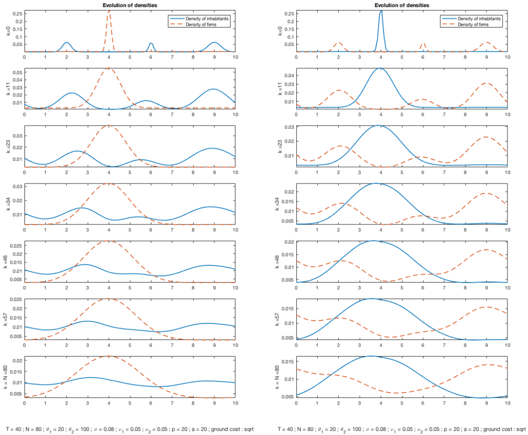
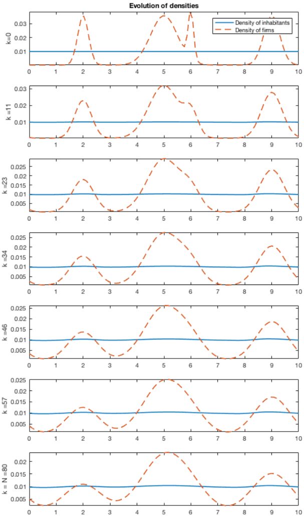


Figure 2: Reversed initial conditions example



T = 40 ; N = 80 ; θ_1 = 150 ; θ_2 = 200 ; σ = 0.1 ; ν_1 = 0.01 ; ν_2 = 0.01 ; p = 50 ; a = 100 ; ground cost : sqrt

Figure 3: A multi-centric city

What to take from the simulations ?

- A very **sparse model** can generate **rich and complex dynamics**
- High sensitivity to commuting cost
- High sensitivity to initial conditions
- Apparently intuitive comparative statics (speed of convergence, agglomeration effects)
- Diverse segregation patterns

  $\rightarrow$ Varying parameters yield the American-style city with a business center and residential suburbs, its inverted ("European") form, a bimodal city in, a near-uniform city with several industrial centers,...

Conclusion

- A general framework
  - → New tools
  - → Well defined
  - → Numerical Method
- Further Applications ?
- Thank you !